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Fundamental Automorphisms of Clifford Algebras and an Extension of Dąbrowski Pin Groups

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Abstract

Double coverings of the orthogonal groups of the real and complex spaces are considered. The relation between discrete transformations of these spaces and fundamental automorphisms of Clifford algebras is established, where an isomorphism between a finite group of the discrete transformations and an automorphism group of the Clifford algebras plays a central role. The complete classification of Dąbrowski groups depending upon signatures of the spaces is given. Two types of Dąbrowski quotient groups are introduced in case of odd-dimensional spaces. Application potentialities of the introduced quotient groups in Physics are discussed.

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1 Introduction

It is known that there are eight double coverings of the orthogonal group $O(p,q)$ [Dąb88, BD89]:

$$\rho^{a,b,c}:\ \mathbf{Pin}^{a,b,c}(p,q)\longrightarrow O(p,q),$$

where $a,b,c\in\{+,-\}$. The group $O(p,q)$ consists of four connected components: identity connected component $O_0(p,q)$, and three components corresponding to parity reversal P, time reversal T, and the combination of these two PT, i.e., $O(p,q)=O_0(p,q)\cup P(O_0(p,q))\cup T(O_0(p,q))\cup PT(O_0(p,q))$. Further, since the four element group (reflection group) $\{1,P,T,PT\}$ is isomorphic to the finite group $\mathbb{Z}_2\otimes\mathbb{Z}_2$ (Gauss-Klein group) [Sal81a, Sal84], then $O(p,q)$ may be represented by a semidirect product $O(p,q)\simeq O_0(p,q)\odot(\mathbb{Z}_2\otimes\mathbb{Z}_2)$. The signs of a,b,c correspond to the signs of the squares of the elements in

$\mathbf{Pin}^{a,b,c}(p,q)$ which cover space reflection, time reversal and a combination of these two: $P^2=a$, $T^2=b$, $(PT)^2=c$. An explicit form of the group $\mathbf{Pin}^{a,b,c}(p,q)$ is given by the following semidirect product

$$\mathbf{Pin}^{a,b,c}(p,q)\simeq\frac{(\mathbf{Spin}_0(p,q)\odot C^{a,b,c})}{\mathbb{Z}_2},$$

where $C^{a,b,c}$ are the four double coverings of $\mathbb{Z}_2\otimes\mathbb{Z}_2$. The all eight double coverings of the orthogonal group $O(p,q)$ are given in the following table:

$a\ b\ c$	$C^{a,b,c}$	Remark
$+\ +\ +$	$\mathbb{Z}_2\otimes\mathbb{Z}_2\otimes\mathbb{Z}_2$	$PT=TP$
$+\ -\ -$	$\mathbb{Z}_2\otimes\mathbb{Z}_4$	$PT=TP$
$-\ +\ -$	$\mathbb{Z}_2\otimes\mathbb{Z}_4$	$PT=TP$
$-\ -\ +$	$\mathbb{Z}_2\otimes\mathbb{Z}_4$	$PT=TP$
$-\ -\ -$	Q_4	$PT=-TP$
$-\ +\ +$	D_4	$PT=-TP$
$+\ -\ +$	D_4	$PT=-TP$
$+\ +\ -$	D_4	$PT=-TP$

Here Q_4 is a quaternion group, and D_4 is a dihedral group. According to [Dąb88] the group $\mathbf{Pin}^{a,b,c}(p,q)$ satisfying to condition $PT=-TP$ is called *Cliffordian*, and respectively *non-Cliffordian* when $PT=TP$.

The **Pin** and **Spin** groups (Clifford-Lipschitz groups) widely used in algebraic topology [BH, Hae56, AtBSh, Kar68, Kar79, KT89], in the definition of pinor and spinor structures on the riemannian manifolds [Mil63, Ger68, Ish78, Wh78, DT86, DP87, LM89, DR89, Cru90, AlCh94, Ch94a, Ch94b, CGT95, AlCh96], spinor bundles [RF90, RO90, FT96, Fr98, FT99], and also have great importance in the theory of the Dirac operator on manifolds [Bau81, Bär91, Tr92, Fr97, Amm98]. The Clifford-Lipschitz groups also intensively used in theoretical physics [CDD82, DW90, FRO90a, RS93, DWGK, Ch97]. In essence, the Dąbrowski group is a 'detailed' (correct to a group of discrete transformations) Clifford-Lipschitz group.

In the present paper the Dąbrowski groups considered in the real and complex spaces $\mathbb{R}^{p,q}$ and $\mathbb{C}^n$, respectively. The finite group $\{1,P,T,PT\}$ is associated with an automorphism group $\{\mathrm{Id},\star,\widetilde{\ },\widetilde{\star}\}$ of the Clifford algebras $C\!\ell_{p,q}$ and $\mathbb{C}_{p+q}$. At first, consideration carried out for even-dimensional algebras $C\!\ell_{p,q}$, $\mathbb{C}_{p+q}$ $(p+q=2m)$ and associated spaces $\mathbb{R}^{p,q}$ and $\mathbb{C}^{p+q}$. A relation between signatures of the Clifford algebras $(p,q)=(\underbrace{++\ldots+}_{p\text{ times}},\underbrace{--\ldots-}_{q\text{ times}})$ and signatures of the Dąbrowski groups (a,b,c) is established in section 4. It is shown that there exist eight non-isomorphic relations between (p,q) and (a,b,c) over the field $\mathbb{F}=\mathbb{R}$ and

only two over the field $\mathbb{F} = \mathbb{C}$. Further, odd-dimensional spaces are considered in section 5, at this point the Clifford algebra is understood as a direct sum of two even-dimensional subalgebras. It is shown (Theorem 11) that in this case there exist two quotient groups $\mathbf{Pin}^b$ and $\mathbf{Pin}^{b,c}$, the latter group exists only over the field $\mathbb{F} = \mathbb{C}$: $\mathbf{Pin}^{b,c}(p+q-1, \mathbb{C})$ if $p+q \equiv 1, 5 \pmod 8$. The set of discrete transformations of the quotient group $\mathbf{Pin}^{b,c}(p+q-1, \mathbb{C})$ does not form a group. It allows to relate this group with some chiral field in Physics. In conclusion, by way of example, the algebra $\mathbb{C}_3$ and the quotient group $\mathbf{Pin}^{b,c}(p+q-1, \mathbb{C})$ are related with a Dirac-Hestenes spinor field [Hest66, Hest90], which has a broad application both in Physics and Geometry [Lou93, RVR93, RRSV95, RSVL, Var99a, Var99b].

2 Algebraic Preliminaries

Clifford algebras play a key role in the definition of the **Pin** groups. Thus, in this section we will consider some basic facts about Clifford algebras which relevant to definition and construction of the **Pin** groups. Let $\mathbb{F}$ be a field of characteristic 0 ($\mathbb{F} = \mathbb{R}$, $\mathbb{F} = \Omega$, $\mathbb{F} = \mathbb{C}$), where Ω is a field of double numbers ($\Omega = \mathbb{R} \oplus \mathbb{R}$), and $\mathbb{R}$, $\mathbb{C}$ are the fields of real and complex numbers, respectively. A Clifford algebra over a field $\mathbb{F}$ is an algebra with 2^n basis elements: $\mathbf{e}_0$ (unit of the algebra), $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ and the products of the one-index elements $\mathbf{e}_{i_1 i_2 \ldots i_k} = \mathbf{e}_{i_1}\mathbf{e}_{i_2}\ldots\mathbf{e}_{i_k}$. Over the field $\mathbb{F} = \mathbb{R}$ the Clifford algebra is denoted as $C\!\ell_{p,q}$, where the indices p, q correspond to the indices of the quadratic form

$$Q = x_1^2 + \ldots + x_p^2 - \ldots - x_{p+q}^2$$

of a vector space V associated with $C\!\ell_{p,q}$. A multiplication law of $C\!\ell_{p,q}$ is defined by the following rule:

$$\mathbf{e}_i^2 = \sigma(q-i)\mathbf{e}_0, \quad \mathbf{e}_i\mathbf{e}_j = -\mathbf{e}_j\mathbf{e}_i, \tag{1}$$

where

$$\sigma(n) = \begin{cases} -1 & \text{if } n \leq 0, \\ +1 & \text{if } n > 0. \end{cases} \tag{2}$$

The square of the volume element $\omega = \mathbf{e}_{12\ldots n}$, $n = p+q$, plays an important role in the theory of Clifford algebras:

$$\omega^2 = \begin{cases} -1 & \text{if } p-q \equiv 1, 2, 5, 6 \pmod 8, \\ +1 & \text{if } p-q \equiv 0, 3, 4, 7 \pmod 8. \end{cases} \tag{3}$$

A center $\mathbf{Z}_{p,q}$ of $C\!\ell_{p,q}$ consists of the unit $\mathbf{e}_0$ and the volume element ω. The element $\omega = \mathbf{e}_{12\ldots n}$ is belong to a center when n is odd. Indeed,

$$\begin{aligned} \mathbf{e}_{12\ldots n}\mathbf{e}_i &= (-1)^{n-i}\sigma(q-i)\mathbf{e}_{12\ldots i-1 i+1\ldots n}, \\ \mathbf{e}_i\mathbf{e}_{12\ldots n} &= (-1)^{i-1}\sigma(q-i)\mathbf{e}_{12\ldots i-1 i+1\ldots n}, \end{aligned}$$

therefore, $\omega \in \mathbf{Z}_{p,q}$ if and only if $n-i \equiv i-1 \pmod 2$, that is, n is odd. Using (3) we have

$$\mathbf{Z}_{p,q} = \begin{cases} 1 & \text{if } p-q \equiv 0,2,4,6 \pmod 8, \\ 1,\omega & \text{if } p-q \equiv 1,3,5,7 \pmod 8. \end{cases} \tag{4}$$

Moreover, when n is odd a center $\mathbf{Z}_{p,q}$ is isomorphic to the fields $\mathbb{C}$ and Ω, respectively:

$$\mathbf{Z}_{p,q} \simeq \begin{cases} \mathbb{R} \oplus i\mathbb{R} & \text{if } p-q \equiv 1,5 \pmod 8, \\ \mathbb{R} \oplus e\mathbb{R} & \text{if } p-q \equiv 3,7 \pmod 8, \end{cases}$$

where e is a double unit ($e^2 = 1$).

Further, let $\mathbb{C}_n = \mathbb{C} \otimes C\!\ell_{p,q}$ and $\Omega_{p,q} = \Omega \otimes C\!\ell_{p,q}$ be the Clifford algebras over the fields $\mathbb{F} = \mathbb{C}$ and $\mathbb{F} = \Omega$, respectively.

Theorem 1. *If $n = p+q$ is odd, then*

$$\begin{aligned} C\!\ell_{p,q} &\simeq \mathbb{C}_{p+q-1} \quad \textit{if } p-q \equiv 1,5 \pmod 8, \\ C\!\ell_{p,q} &\simeq \Omega_{p-1,q} \\ &\simeq \Omega_{p,q-1} \quad \textit{if } p-q \equiv 3,7 \pmod 8. \end{aligned}$$

Proof. The structure of $C\!\ell_{p,q}$ allows to identify the Clifford algebras over the different fields. Indeed, transitions $C\!\ell_{p-1,q} \rightarrow C\!\ell_{p,q}$, $C\!\ell_{p,q-1} \rightarrow C\!\ell_{p,q}$ may be represented as transitions from the real coordinates in $C\!\ell_{p-1,q}$, $C\!\ell_{p,q-1}$ to complex coordinates of the form $a + \omega b$, where ω is an additional basis element $\mathbf{e}_{12\ldots n}$ (volume element). Since $n = p+q$ is odd, then the volume element ω in accordance with (4) belongs to $\mathbf{Z}_{p,q}$. Therefore, we can to identify it with imaginary unit i if $p-q \equiv 1,5 \pmod 8$ and with a double unit e if $p-q \equiv 3,7 \pmod 8$. The general element of the algebra $C\!\ell_{p,q}$ has a form $\mathcal{A} = \mathcal{A}' + \omega\mathcal{A}'$, where $\mathcal{A}'$ is a general element of the algebras $C\!\ell_{p-1,q}$, $C\!\ell_{p,q-1}$. $\square$

Example. Let us consider the algebra $C\!\ell_{0,3}$. According to the theorem 1 we have $C\!\ell_{0,3} \simeq \Omega_{0,2}$, where $\Omega_{0,2}$ is an algebra of elliptic biquaternions (it is a first so-called Grassmann's extensive algebra introduced by Clifford in 1878 [Cliff]). Since $\Omega = \mathbb{R} \oplus \mathbb{R}$ and $\Omega_{p,q} = \Omega \otimes C\!\ell_{p,q}$, we have $C\!\ell_{0,3} \simeq C\!\ell_{0,2} \oplus C\!\ell_{0,2} \simeq \mathbb{H} \oplus \mathbb{H}$, where $\mathbb{H}$ is a quaternion algebra.

Generalizing this example we obtain

$$\begin{aligned} C\!\ell_{p,q} &\simeq C\!\ell_{p-1,q} \oplus C\!\ell_{p-1,q} \\ &\simeq C\!\ell_{p,q-1} \oplus C\!\ell_{p,q-1} \quad \text{if } p-q \equiv 3,7 \pmod 8. \end{aligned} \tag{5}$$

Over the field $\mathbb{F} = \mathbb{C}$ there is the analogous result [Ras55].

Theorem 2. *When* $p+q \equiv 1, 3, 5, 7 \pmod 8$ *the Clifford algebra over the field* $\mathbb{F}=\mathbb{C}$ *decomposes into a direct sum of two subalgebras:*

$$\mathbb{C}_{p+q} \simeq \mathbb{C}_{p+q-1} \oplus \mathbb{C}_{p+q-1}.$$

In Clifford algebra $C\!\ell_{p,q}$ there exist four fundamental automorphisms [Sch49, Ras55]:

1) An automorphism $\mathcal{A} \to \mathcal{A}$.
This automorphism, obviously, is an identical automorphism of the algebra $C\!\ell_{p,q}$, $\mathcal{A}$ is an arbitrary element of $C\!\ell_{p,q}$.

2) An automorphism $\mathcal{A} \to \mathcal{A}^\star$.
In more details, for arbitrary element $\mathcal{A} \in C\!\ell_{p,q}$ there exists a decomposition

$$\mathcal{A} = \mathcal{A}' + \mathcal{A}'',$$

where $\mathcal{A}'$ is an element consisting of homogeneous odd elements, and $\mathcal{A}''$ is an element consisting of homogeneous even elements, respectively. Then the automorphism $\mathcal{A} \to \mathcal{A}^\star$ is such that the element $\mathcal{A}''$ is not changed, and the element $\mathcal{A}'$ changes sign:

$$\mathcal{A}^\star = -\mathcal{A}' + \mathcal{A}''.$$

If $\mathcal{A}$ is a homogeneous element, then

$$\mathcal{A}^\star = (-1)^k \mathcal{A}, \tag{6}$$

where k is a degree of element. It is easy to see that the automorphism $\mathcal{A} \to \mathcal{A}^\star$ may be expressed via the volume element ω:

$$\mathcal{A}^\star = \omega \mathcal{A} \omega^{-1}, \tag{7}$$

where $\omega^{-1} = (-1)^{\frac{n(n-1)}{2}}\omega$, $\mathcal{A}$ is an arbitrary element of $C\!\ell_{p,q}$. When k is odd, for the basis elements $\mathbf{e}_{i_1 i_2 \ldots i_k}$ the sign changes, and when k is even the sign is not changed. Over the field $\mathbb{F}=\mathbb{C}$ we can multiply ω by $\varepsilon = \pm i^{\frac{n(n-1)}{2}}$, then the equality (7) is not changed. At this point we have always $(\varepsilon\omega)^2 = 1$. Therefore,

$$\mathcal{A}^\star = (\varepsilon\omega)\mathcal{A}(\varepsilon\omega). \tag{8}$$

3) An antiautomorphism $\mathcal{A} \to \widetilde{\mathcal{A}}$.
The antiautomorphism $\mathcal{A} \to \widetilde{\mathcal{A}}$ is a reversion of the element $\mathcal{A}$, that is, the substitution of the each basis element $\mathbf{e}_{i_1 i_2 \ldots i_k} \in \mathcal{A}$ by the element $\mathbf{e}_{i_k i_{k-1} \ldots i_1}$:

$$\mathbf{e}_{i_k i_{k-1} \ldots i_1} = (-1)^{\frac{k(k-1)}{2}} \mathbf{e}_{i_1 i_2 \ldots i_k}.$$

Therefore, for any $\mathcal{A} \in C\!\ell_{p,q}$ we have

$$\widetilde{\mathcal{A}} = (-1)^{\frac{k(k-1)}{2}} \mathcal{A}. \tag{9}$$

4) An antiautomorphism $\mathcal{A} \to \widetilde{\mathcal{A}^\star}$.

This antiautomorphism is a composition of the antiautomorphism $\mathcal{A} \to \widetilde{\mathcal{A}}$ with the automorphism $\mathcal{A} \to \mathcal{A}^\star$. In the case of homogeneous element from formulae (6) and (9) it follows

$$\widetilde{\mathcal{A}^\star} = (-1)^{\frac{k(k+1)}{2}}\mathcal{A}. \tag{10}$$

It is obvious that $\widetilde{\widetilde{\mathcal{A}}} = \mathcal{A}$, $(\mathcal{A}^\star)^\star = \mathcal{A}$, and $(\widetilde{\widetilde{\mathcal{A}^\star})^\star} = \mathcal{A}$.

The Lipschitz group $\mathbf{\Gamma}_{p,q}$, also called the Clifford group, introduced by Lipschitz in 1886 [Lips], may be defined as the subgroup of invertible elements s of the algebra $C\!\ell_{p,q}$:

$$\mathbf{\Gamma}_{p,q} = \left\{ s \in C\!\ell^+_{p,q} \cup C\!\ell^-_{p,q} \mid \forall x \in \mathbb{R}^{p,q},\ s\mathbf{x}s^{-1} \in \mathbb{R}^{p,q} \right\}.$$

The set $\mathbf{\Gamma}^+_{p,q} = \mathbf{\Gamma}_{p,q} \cap C\!\ell^+_{p,q}$ is called *special Lipschitz group* [Che54].

Let $N:\ C\!\ell_{p,q} \to C\!\ell_{p,q},\ N(\mathbf{x}) = \mathbf{x}\widetilde{\mathbf{x}}$. If $\mathbf{x} \in \mathbb{R}^{p,q}$, then $N(\mathbf{x}) = \mathbf{x}(-\mathbf{x}) = -\mathbf{x}^2 = -Q(\mathbf{x})$. Further, the group $\mathbf{\Gamma}_{p,q}$ has a subgroup

$$\mathbf{Pin}(p,q) = \{ s \in \mathbf{\Gamma}_{p,q} \mid N(s) = \pm 1 \}. \tag{11}$$

Analogously, *a spinor group* $\mathbf{Spin}(p,q)$ is defined by the set

$$\mathbf{Spin}(p,q) = \left\{ s \in \mathbf{\Gamma}^+_{p,q} \mid N(s) = \pm 1 \right\}. \tag{12}$$

It is obvious that

$$\mathbf{Spin}(p,q) = \mathbf{Pin}(p,q) \cap C\!\ell^+_{p,q}.$$

The group $\mathbf{Spin}(p,q)$ contains a subgroup

$$\mathbf{Spin}_+(p,q) = \{ s \in \mathbf{Spin}(p,q) \mid N(s) = 1 \}. \tag{13}$$

It is easy to see that the groups $O(p,q)$, $SO(p,q)$ and $SO_+(p,q)$ are isomorphic correspondngly to the following quotient groups

$$\begin{aligned} O(p,q) &\simeq \mathbf{Pin}(p,q)/\mathbb{Z}_2, \\ SO(p,q) &\simeq \mathbf{Spin}(p,q)/\mathbb{Z}_2, \\ SO_+(p,q) &\simeq \mathbf{Spin}_+(p,q)/\mathbb{Z}_2, \end{aligned}$$

where a kernel $\mathbb{Z}_2 = \{1, -1\}$. Thus, the groups $\mathbf{Pin}(p,q)$, $\mathbf{Spin}(p,q)$ and $\mathbf{Spin}_+(p,q)$ are the double coverings of the groups $O(p,q)$, $SO(p,q)$ and $SO_+(p,q)$, respectively.

Further, since $C\!\ell^+_{p,q} \simeq C\!\ell^+_{q,p}$, then

$$\mathbf{Spin}(p,q) \simeq \mathbf{Spin}(q,p).$$

In contrast with this, the groups $\mathbf{Pin}(p,q)$ and $\mathbf{Pin}(q,p)$ are non-isomorphic. Denote $\mathbf{Spin}(n) = \mathbf{Spin}(n,0) \simeq \mathbf{Spin}(0,n)$.

Theorem 3 ([Cor84])**.** *The spinor groups*

$$\mathbf{Spin}(2),\ \ \mathbf{Spin}(3),\ \ \mathbf{Spin}(4),\ \ \mathbf{Spin}(5),\ \ \mathbf{Spin}(6)$$

are isomorphic to the unitary groups

$$U(1),\ \ Sp(1)\sim SU(2),\ \ SU(2)\times SU(2),\ \ Sp(2),\ \ SU(4).$$

In accordance with Theorem 1 and decompositions (5) over the field $\mathbb{F}=\mathbb{R}$ the algebra $C\!\ell_{p,q}$ is isomorphic to a direct sum of two mutually annihilating simple ideals $\frac{1}{2}(1\pm\omega)C\!\ell_{p,q}$: $C\!\ell_{p,q}\simeq\frac{1}{2}(1+\omega)C\!\ell_{p,q}\oplus\frac{1}{2}(1-\omega)C\!\ell_{p,q}$, where $\omega=\mathbf{e}_{12\ldots p+q}$, $p-q\equiv 3,7 \pmod 8$. At this point, each ideal is isomorpic to $C\!\ell_{p-1,q}$ or $C\!\ell_{p,q-1}$. Therefore, for the Clifford-Lipschitz groups we have the following isomorphisms

$$\begin{aligned}\mathbf{Pin}(p,q) &\simeq \mathbf{Pin}(p-1,q)\cup\mathbf{Pin}(p-1,q)\\ &\simeq \mathbf{Pin}(p,q-1)\cup\mathbf{Pin}(p,q-1).\end{aligned}\tag{14}$$

Or, since $C\!\ell_{p-1,q}\simeq C\!\ell^+_{p,q}\subset C\!\ell_{p,q}$, then according to (12)

$$\mathbf{Pin}(p,q)\simeq\mathbf{Spin}(p,q)\cup\mathbf{Spin}(p,q)$$

if $p-q\equiv 3,7 \pmod 8$.

Further, when $p-q\equiv 1,5 \pmod 8$ from Theorem 1 it follows that $C\!\ell_{p,q}$ is isomorphic to a complex algebra $\mathbb{C}_{p+q-1}$. Therefore, for the **Pin** groups we obtain

$$\begin{aligned}\mathbf{Pin}(p,q) &\simeq \mathbf{Pin}(p-1,q)\cup\mathbf{e}_{12\ldots p+q}\mathbf{Pin}(p-1,q)\\ &\simeq \mathbf{Pin}(p,q-1)\cup\mathbf{e}_{12\ldots p+q}\mathbf{Pin}(p,q-1)\end{aligned}\tag{15}$$

if $p-q\equiv 1,5 \pmod 8$ and correspondingly

$$\mathbf{Pin}(p,q)\simeq\mathbf{Spin}(p,q)\cup\mathbf{e}_{12\ldots p+q}\mathbf{Spin}(p,q).\tag{16}$$

In case $p-q\equiv 3,7 \pmod 8$ we have isomorphisms which are analoguos to (15)-(16), since $\omega C\!\ell_{p,q}\sim C\!\ell_{p,q}$. Generalizing we obtain the following

Theorem 4. *Let* $\mathbf{Pin}(p,q)$ *and* $\mathbf{Spin}(p,q)$ *be the Clifford-Lipschitz groups of the invertible elements of the algebras* $C\!\ell_{p,q}$ *with odd dimensionality,* $p-q\equiv 1,3,5,7 \pmod 8$. *Then*

$$\begin{aligned}\mathbf{Pin}(p,q) &\simeq \mathbf{Pin}(p-1,q)\cup\omega\,\mathbf{Pin}(p-1,q)\\ &\simeq \mathbf{Pin}(p,q-1)\cup\omega\,\mathbf{Pin}(p,q-1)\end{aligned}$$

and

$$\mathbf{Pin}(p,q)\simeq\mathbf{Spin}(p,q)\cup\omega\,\mathbf{Spin}(p,q),$$

where $\omega=\mathbf{e}_{12\ldots p+q}$ *is a volume element of* $C\!\ell_{p,q}$.

In case of low dimensionalities from Theorem 3 and Theorem 4 it immediately follows

Theorem 5. *For $p+q \leq 5$ and $p-q \equiv 3,5 \pmod 8$,*

$$\begin{aligned}
\mathbf{Pin}(3,0) &\simeq SU(2) \cup iSU(2),\\
\mathbf{Pin}(0,3) &\simeq SU(2) \cup eSU(2),\\
\mathbf{Pin}(5,0) &\simeq Sp(2) \cup eSp(2),\\
\mathbf{Pin}(0,5) &\simeq Sp(2) \cup iSp(2).
\end{aligned}$$

Proof. Indeed, in accordance with Theorem 4 $\mathbf{Pin}(3,0) \simeq \mathbf{Spin}(3) \cup \mathbf{e}_{123}\,\mathbf{Spin}(3)$. Further, from Theorem 3 we have $\mathbf{Spin}(3) \simeq SU(2)$, and a square of the element $\omega = \mathbf{e}_{123}$ is equal to -1, therefore $\omega \sim i$. Thus, $\mathbf{Pin}(3,0) \simeq SU(2) \cup iSU(2)$. For the group $\mathbf{Pin}(0,3)$ a square of ω is equal to $+1$, therefore $\mathbf{Pin}(0,3) \simeq SU(2) \cup eSU(2)$, e is a double unit. As expected, $\mathbf{Pin}(3,0) \not\simeq \mathbf{Pin}(0,3)$. The isomorphisms for the groups $\mathbf{Pin}(5,0)$ and $\mathbf{Pin}(0,5)$ are analogously proved. $\square$

Further, let E be a vector space, then a homomorphism

$$\rho:\ \mathcal{C}\!\ell_{p,q} \longrightarrow \operatorname{End} E,$$

which maps the unit element of the algebra $\mathcal{C}\!\ell_{p,q}$ to Id_E, is called *a representation* of $\mathcal{C}\!\ell_{p,q}$ in E ($\operatorname{End} E$ is an endomorphism algebra of the space E). The dimensionality of E is called a degree of the representation. The addition in E together with the mapping $\mathcal{C}\!\ell_{p,q} \times E \rightarrow E$, $(a,x) \mapsto \rho(a)x$, $a \in \mathcal{C}\!\ell_{p,q}$, $x \in E$, turns E in $\mathcal{C}\!\ell_{p,q}$-module, *a representation module.* The representation ρ is faithful if its kernel is zero, that is, $\rho(a)x = 0$, $\forall x \in E \Rightarrow a = 0$. If the representation ρ has only two invariant subspaces E and $\{0\}$, then ρ is said to be simple or irreducible. On the contrary case, ρ is said to be semi-simple, that is, it is a direct sum of simple modules, and in this case E is a direct sum of subspaces which are globally invariant under $\rho(a)$, $\forall a \in \mathcal{C}\!\ell_{p,q}$. The representation ρ of $\mathcal{C}\!\ell_{p,q}$ induces a representation of the group $\mathbf{Pin}(p,q)$ which we will denote by the same symbol ρ, and also induces a representation of the group $\mathbf{Spin}(p,q)$ which we will denote by $\Delta_{p,q}$. In so doing, we have the following [Che54, Ras55]

Theorem 6. *If $p+q = 2m$ and $p-q \equiv 0,2,4,6 \pmod 8$, then*

$$\mathcal{C}\!\ell_{p,q} \simeq \operatorname{End}_{\mathbb{F}}(I_{p,q}) \simeq \mathsf{M}_{2^m}(\mathbb{F})$$

and

$$\mathcal{C}\!\ell_{p,q} \simeq \operatorname{End}_{\mathbb{F}}(I_{p,q}) \simeq \mathsf{M}_{2^m}(\mathbb{F}) \oplus \mathsf{M}_{2^m}(\mathbb{F})$$

if $p+q = 2m+1$ and $p-q \equiv 1,3,5,7 \pmod 8$, where $\mathbb{F} = \mathbb{R}, \mathbb{C}, \Omega, \mathbb{H}$, $I_{p,q}$ is a minimal left ideal of $\mathcal{C}\!\ell_{p,q}$, $\operatorname{End}_{\mathbb{F}}(I_{p,q})$ is an algebra of linear transformations in $I_{p,q}$ over the field $\mathbb{F}$, $\mathsf{M}_{2^m}(\mathbb{F})$ is a matrix algebra.

Let us consider matrix representations of the fundamental automorphisms of $C\!\ell_{p,q}$ over the field $\mathbb{F}$ when $p+q$ is even [Sch49, Ras55]. We start with the antiautomorphism $\mathcal{A} \rightarrow \widetilde{\mathcal{A}}$. In accordance with Theorem 6 in the matrix representation the antiautomorphism $\mathcal{A} \rightarrow \widetilde{\mathcal{A}}$ corresponds to an antiautomorphism of the matrix algebra $\mathsf{M}_{2^m}(\mathbb{F})$:

$$\mathsf{A} \longrightarrow \mathsf{A}^T,$$

in virtue of the well-known relation $(\mathsf{AB})^T = \mathsf{B}^T\mathsf{A}^T$, where T is a symbol of transposition. On the other hand, in the matrix representation of the elements $\mathcal{A} \in C\!\ell_{p,q}$ for the antiautomorphism $\mathcal{A} \rightarrow \widetilde{\mathcal{A}}$ we have

$$\mathsf{A} \longrightarrow \widetilde{\mathsf{A}}.$$

The composition of the two antiautomorphisms $\mathsf{A}^T \rightarrow \mathsf{A} \rightarrow \widetilde{\mathsf{A}}$ gives an automorphism $\mathsf{A}^T \rightarrow \widetilde{\mathsf{A}}$ which is an internal automorphism of the algebra $M_{2^m}(\mathbb{F})$:

$$\widetilde{\mathsf{A}} = \mathsf{E}\mathsf{A}^T\mathsf{E}^{-1}, \tag{17}$$

where E is a matrix, by means of which the antiautomorphism $\mathcal{A} \rightarrow \widetilde{\mathcal{A}}$ is expressed in the matrix representation of the algebra $C\!\ell_{p,q}$.

Further, for the automorphism $\mathcal{A} \rightarrow \mathcal{A}^\star$, defined by the formula (7), in the matrix representation we have

$$\mathsf{A}^\star = \mathsf{W}\mathsf{A}\mathsf{W}^{-1}, \tag{18}$$

where A is a matrix representing an arbitrary element of $C\!\ell_{p,q}$, W is a matrix of the volume element $\omega = \mathbf{e}_{12\ldots n}$. Over the field $\mathbb{F} = \mathbb{C}$ we can multiply W by the factor $\varepsilon = \pm i^{\frac{(p+q)(p+q-1)}{2}}$, then $(\varepsilon\mathsf{W})^2 = 1$. Therefore, the relation (18) may be rewritten in the form

$$\mathsf{A}^\star = \mathsf{W}'\mathsf{A}\mathsf{W}', \tag{19}$$

where $\mathsf{W}' = \varepsilon\mathsf{W}$.

Finally, for the antiautomorphism $\mathcal{A} \rightarrow \widetilde{\mathcal{A}^\star}$, which is the composition of the antiautomorphism $\mathcal{A} \rightarrow \widetilde{\mathcal{A}}$ with the automorphism $\mathcal{A} \rightarrow \mathcal{A}^\star$, using (17) and (19) we obtain a following expression

$$\widetilde{\mathsf{A}^\star} = \mathsf{W}'\mathsf{E}\mathsf{A}^T\mathsf{E}^{-1}\mathsf{W}',$$

or

$$\widetilde{\mathsf{A}^\star} = (\mathsf{E}\mathsf{W}'^T)\mathsf{A}^T(\mathsf{E}\mathsf{W}'^T)^{-1}. \tag{20}$$

Denoting $\mathsf{E}\mathsf{W}'^T = \mathsf{C}$, where C is a matrix representation the antiautomorphism $\mathcal{A} \rightarrow \widetilde{\mathcal{A}^\star}$, and substituting into (20) we obtain definitely

$$\widetilde{\mathsf{A}^\star} = \mathsf{C}\mathsf{A}^T\mathsf{C}^{-1}. \tag{21}$$

Example. Let consider matrix representations of the fundamental automorphisms of the Dirac algebra $\mathcal{C}\!\ell_{4,1}$. In virtue of Theorem 1 there is an isomorphism $\mathcal{C}\!\ell_{4,1} \simeq \mathbb{C}_4$, and therefore $\mathcal{C}\!\ell_{4,1} \simeq \mathbb{C}_4 \simeq \mathsf{M}_4(\mathbb{C})$. In the capacity of the matrix representations of the units $\mathbf{e}_i \in \mathbb{C}_4$ ($i = 1, 2, 3, 4$) we take the well-known Dirac γ-matrices (so-called canonical representation):

$$\gamma_1 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},$$

$$\gamma_3 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, \quad \gamma_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{22}$$

γ-matrices form the only one basis from the set of isomorphic matrix basises of $\mathcal{C}\!\ell_{4,1} \simeq \mathbb{C}_4$. In the basis (22) the element $\omega = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\mathbf{e}_4$ is represented by a matrix $\mathsf{W} = \gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4$. Since $\gamma_5^2 = 1$, then $\varepsilon = 1$ ($\mathsf{W}' = \mathsf{W}$) and the matrix

$$\mathsf{W}^T = \mathsf{W} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \tag{23}$$

in accordance with (19) is a matrix of the automorphism $\mathcal{A} \rightarrow \mathcal{A}^\star$. Further, in the matrix representation the antiautomorphism $\mathcal{A} \rightarrow \widetilde{\mathcal{A}}$ is defined by the transformation $\widetilde{\mathsf{A}} = \mathsf{E}\mathsf{A}^T\mathsf{E}^{-1}$. For the γ-matrices we have $\gamma_1^T = -\gamma_1$, $\gamma_2^T = \gamma_2$, $\gamma_3^T = -\gamma_3$, $\gamma_4^T = \gamma_4$. Further,

$$\gamma_1 = -\mathsf{E}\gamma_1\mathsf{E}^{-1}, \quad \gamma_2 = \mathsf{E}\gamma_2\mathsf{E}^{-1},$$
$$\gamma_3 = -\mathsf{E}\gamma_3\mathsf{E}^{-1}, \quad \gamma_4 = \mathsf{E}\gamma_4\mathsf{E}^{-1}.$$

It is easy to verify that a matrix $\mathsf{E} = \gamma_1\gamma_3$ satisfies the latter relations and, therefore, the antiautomorphism $\mathcal{A} \rightarrow \widetilde{\mathcal{A}}$ in the basis (22) is defined by the matrix

$$\mathsf{E} = \gamma_1\gamma_3 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{24}$$

Finally, for the matrix $\mathsf{C} = \mathsf{E}\mathsf{W}^T$ of the antiautomorphism $\mathcal{A} \rightarrow \widetilde{\mathcal{A}^\star}$ from (23) and (24) in accordance with (21) we obtain

$$\mathsf{C} = \mathsf{E}\mathsf{W}^T = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}. \tag{25}$$

3 Fundamental Automorphisms of Odd-dimensional Clifford Algebras

Let us consider the fundamental automorphisms of the algebras $C\!\ell_{p,q}$ and $\mathbb{C}_{p+q}$, where $p - q \equiv 1, 3, 5, 7 \pmod 8$, $p + q = 2m + 1$. In accordance with (5) and Theorem 2 the algebras $C\!\ell_{p,q}$ and $\mathbb{C}_{p+q}$ are isomorphic to direct sums of two algebras with the even dimensionality if correspondingly $p - q \equiv 3, 7 \pmod 8$ and $p + q \equiv 1, 3, 5, 7 \pmod 8$. Therefore, matrix representations of $C\!\ell_{p,q}$, $\mathbb{C}_{p+q}$ are isomorphic to the direct sums of complete matrix algebras $\mathsf{M}_{2^m}(\mathbb{F}) \oplus \mathsf{M}_{2^m}(\mathbb{F})$, here $\mathbb{F} = \mathbb{R}$, $\mathbb{F} = \mathbb{C}$. On the other hand, there exists an homomorphic mapping of $C\!\ell_{p,q}$ and $\mathbb{C}_{p+q}$ into one full matrix algebra $\mathsf{M}_{2^m}(\mathbb{F})$ with preservation of addition, multiplication and multiplication by the number. Besides, in the case of $\mathbb{F} = \mathbb{R}$ and $p - q \equiv 1, 5 \pmod 8$ the algebra $C\!\ell_{p,q}$ is isomorphic to the full matrix algebra $\mathsf{M}_{2^m}(\mathbb{C})$ (Theorem 1), therefore, representations of the fundamental automorphisms of this algebra may be realized by means of $\mathsf{M}_{2^m}(\mathbb{C})$.

Theorem 7. *If $p + q = 2m + 1$, then the following homomorphisms take place*
1) $\mathbb{F} = \mathbb{R}$

$$\epsilon : \quad C\!\ell_{p,q} \longrightarrow \mathsf{M}_{2^m}(\mathbb{R}) \quad \textit{if } \ p - q \equiv 3, 7 \pmod 8.$$

2) $\mathbb{F} = \mathbb{C}$

$$\epsilon' : \quad \mathbb{C}_{p+q} \longrightarrow \mathsf{M}_{2^m}(\mathbb{C}) \quad \textit{if } \ p + q \equiv 1, 3, 5, 7 \pmod 8.$$

Proof. We start the proof with a more general case of $\mathbb{F} = \mathbb{C}$. According to (4) the volume element ω belongs to a center of $\mathbb{C}_n$ $(n = p+q)$, therefore, ω commutes with all basis elements of this algebra and $(\varepsilon\omega)^2 = 1$. Further, recalling that a vector complex space $\mathbb{C}^n$ is associated with the algebra $\mathbb{C}_n$, we see that basis vectors $\{e_1, e_2, \dots, e_n\}$ generate a subspace $C_n \subset C_{n+1}$. Thus, the algebra $\mathbb{C}_n$ in C_n is a subalgebra of $\mathbb{C}_{n+1}$ and consists of the elements which does not contain the element $\mathbf{e}_{n+1}$. A decomposition of the each element $\mathcal{A} \in \mathbb{C}_{n+1}$ may be written in the form

$$\mathcal{A} = \mathcal{A}^1 + \mathcal{A}^0,$$

where $\mathcal{A}^0$ is a set of all elements which contain $\mathbf{e}_{n+1}$, and $\mathcal{A}^1$ is a set of all elements which does not contain $\mathbf{e}_{n+1}$, therefore $\mathcal{A}^1 \in \mathbb{C}_n$. If multiply $\mathcal{A}^0$ by $\varepsilon\omega$, then the elements $\mathbf{e}_{n+1}$ are mutually annihilate, therefore $\varepsilon\omega\mathcal{A}^0 \in \mathbb{C}_n$. Denoting $\mathcal{A}^2 = \varepsilon\omega\mathcal{A}^0$ and taking into account $(\varepsilon\omega)^2 = 1$ we obtain

$$\mathcal{A} = \mathcal{A}^1 + \varepsilon\omega\mathcal{A}^2,$$

where $\mathcal{A}^1$, $\mathcal{A}^2 \in \mathbb{C}_n$. Consider now an homomorphism $\epsilon : \mathbb{C}_{n+1} \rightarrow \mathbb{C}_n$, an action of which is defined by the following law

$$\epsilon : \mathcal{A}^1 + \varepsilon\omega\mathcal{A}^2 \longrightarrow \mathcal{A}^1 + \mathcal{A}^2. \tag{26}$$

Obviously, at this point the all operations (addition, multiplication, and multiplication by the number) are preserved. Indeed, let

$$\mathcal{A} = \mathcal{A}^1 + \varepsilon\omega\mathcal{A}^2, \quad \mathcal{B} = \mathcal{B}^1 + \varepsilon\omega\mathcal{B}^2,$$

then in virtue of $(\varepsilon\omega)^2 = 1$ and commutativity of ω with all elements, we have for multiplication

$$\mathcal{A}\mathcal{B} = (\mathcal{A}^1\mathcal{B}^1 + \mathcal{A}^2\mathcal{B}^2) + \varepsilon\omega(\mathcal{A}^1\mathcal{B}^2 + \mathcal{A}^2\mathcal{B}^1) \xrightarrow{\epsilon} \\ (\mathcal{A}^1\mathcal{B}^1 + \mathcal{A}^2\mathcal{B}^2) + (\mathcal{A}^1\mathcal{B}^2 + \mathcal{A}^2\mathcal{B}^1) = (\mathcal{A}^1 + \mathcal{A}^2)(\mathcal{B}^1 + \mathcal{B}^2).$$

that is, the image of product equals to the product of factor images in the same order.

In the particular case of $\mathcal{A} = \varepsilon\omega$ we have $\mathcal{A}^1 = 0$ and $\mathcal{A}^2 = 1$, therefore

$$\varepsilon\omega \longrightarrow 1.$$

Thus, a kernel of the homomorphism ϵ consists of all elements of the form $\mathcal{A}^1 - \varepsilon\omega\mathcal{A}^1$, which under action of ϵ are mapped into zero. It is clear that $\operatorname{Ker}\epsilon = \{\mathcal{A}^1 - \varepsilon\omega\mathcal{A}^1\}$ is a subalgebra of $\mathbb{C}_{n+1}$. Moreover, the kernel of ϵ is a bilateral ideal of $\mathbb{C}_{n+1}$. Therefore, the algebra $\mathbb{C}_n$, which we obtain in the result of the mapping $\epsilon : \mathbb{C}_{n+1} \longrightarrow \mathbb{C}_n$, is *a quotient algebra*

$${}^{\epsilon}\mathbb{C}_n \simeq \mathbb{C}_{n+1}/\operatorname{Ker}\epsilon.$$

Further, since the algebra $\mathbb{C}_n$ $(n = 2m)$ is isomorphic to the full matrix algebra $\mathsf{M}_{2^m}(\mathbb{C})$, then in virtue of $\epsilon : \mathbb{C}_{n+1} \longrightarrow \mathbb{C}_n \subset \mathbb{C}_{n+1}$ we obtain an homomorphic mapping of $\mathbb{C}_{n+1}$ onto the matrix algebra $\mathsf{M}_{2^m}(\mathbb{C})$.

The homomorphism $\epsilon : \mathcal{C}\!\ell_{p,q} \rightarrow \mathsf{M}_{2^m}(\mathbb{R})$ is analogously proved. In this case a quotient algebra has a form

$${}^{\epsilon}\mathcal{C}\!\ell_{p,q} \simeq \mathcal{C}\!\ell_{p+1,q}/\operatorname{Ker}\epsilon$$

or

$${}^{\epsilon}\mathcal{C}\!\ell_{p,q} \simeq \mathcal{C}\!\ell_{p,q+1}/\operatorname{Ker}\epsilon,$$

where $\operatorname{Ker}\epsilon = \{\mathcal{A}^1 - \omega\mathcal{A}^1\}$, since in accordance with (3) at $p - q \equiv 3, 7 \pmod 8$ we have $\omega^2 = 1$, therefore $\varepsilon = 1$. $\square$

Let us consider the form which the fundamental automorphisms of $\mathbb{C}_{n+1}$ take after the homomorphic mapping $\epsilon : \mathbb{C}_{n+1} \to \mathbb{C}_n \subset \mathbb{C}_{n+1}$. First of all, for the antiautomorphism $\mathcal{A} \to \widetilde{\mathcal{A}}$ it is necessary that elements $\mathcal{A}$, $\mathcal{B}$, $\ldots \in \mathbb{C}_{n+1}$, which are mapped into one and the same element $\mathcal{D} \in \mathbb{C}_n$ (a kernel of the homomorphism ϵ if $\mathcal{D} = 0$) after the transformation $\mathcal{A} \to \widetilde{\mathcal{A}}$ are must converted to the elements $\widetilde{\mathcal{A}}$, $\widetilde{\mathcal{B}}$, $\ldots \in \mathbb{C}_{n+1}$, which are also mapped into one and the same element $\widetilde{\mathcal{D}} \in \mathbb{C}_n$. Otherwise, the transformation $\mathcal{A} \to \widetilde{\mathcal{A}}$ is not transferred from $\mathbb{C}_{n+1}$ into $\mathbb{C}_n$ as an unambiguous transformation. In particular, it is necessary in order that $\widetilde{\varepsilon\omega} = \varepsilon\omega$, since 1 and element $\varepsilon\omega$ under action of the homomorphism ϵ are equally mapped into the unit, then $\widetilde{1}$ and $\widetilde{\varepsilon\omega}$ are also must be mapped into one and the same element in $\mathbb{C}_n$, but $\widetilde{1} \to 1$, and $\widetilde{\varepsilon\omega} \to \pm 1$ (in virtue of the formula (9)). Therefore we must assume

$$\widetilde{\varepsilon\omega} = \varepsilon\omega. \tag{27}$$

The condition (27) is sufficient for the transfer of the antiautomorphism $\mathcal{A} \to \widetilde{\mathcal{A}}$ from $\mathbb{C}_{n+1}$ into $\mathbb{C}_n$. Indeed, in this case we have

$$\mathcal{A}^1 - \mathcal{A}^1\varepsilon\omega \longrightarrow \widetilde{\mathcal{A}^1} - \widetilde{\varepsilon\omega}\widetilde{\mathcal{A}^1} = \widetilde{\mathcal{A}^1} - \varepsilon\omega\widetilde{\mathcal{A}^1}.$$

Therefore, the elements of the form $\mathcal{A}^1 - \mathcal{A}^1\varepsilon\omega$ (composing, as known, the kernel of ϵ) under action of the transformation $\mathcal{A} \to \widetilde{\mathcal{A}}$ are converted to the elements of the same form.

The analogous conditions take place for other fundamental automorphisms. However, for the automorphism $\mathcal{A} \to \mathcal{A}^\star$ a condition $(\varepsilon\omega)^\star = \varepsilon\omega$ is not valid, since ω is odd and in accordance with (6) we have

$$\omega^\star = -\omega. \tag{28}$$

Thus, the automorphism $\mathcal{A} \to \mathcal{A}^\star$ is not transferred from $\mathbb{C}_{n+1}$ into $\mathbb{C}_n$.

Let us return to the antiautomorphism $\mathcal{A} \to \widetilde{\mathcal{A}}$ and let consider in more details necessary conditions for the transfer of this transformation from $\mathbb{C}_{n+1}$ to $\mathbb{C}_n$. First of all, the factor ε depending upon the condition $(\varepsilon\omega)^2 = 1$ and the square of the element $\omega = \mathbf{e}_{12\ldots n+1}$ takes the following values

$$\varepsilon = \begin{cases} 1 & \text{if } p - q \equiv 3, 7 \pmod{8}, \\ i & \text{if } p - q \equiv 1, 5 \pmod{8}. \end{cases} \tag{29}$$

Further, in accordance with (9) for the transformation $\omega \to \widetilde{\omega}$ we obtain

$$\widetilde{\omega} = \begin{cases} \omega & \text{if } p - q \equiv 3, 7 \pmod{8}, \\ -\omega & \text{if } p - q \equiv 1, 5 \pmod{8}. \end{cases} \tag{30}$$

Therefore, for the algebras over the field $\mathbb{F}=\mathbb{R}$ the antiautomorphism $\mathcal{A}\rightarrow\widetilde{\mathcal{A}}$ is transfered at the mappings $C\!\ell_{p,q}\rightarrow C\!\ell_{p-1,q}$, $C\!\ell_{p,q}\rightarrow C\!\ell_{p,q-1}$, where $p-q\equiv 3,7 \pmod 8$. Over the field $\mathbb{F}=\mathbb{C}$ the antiautomorphism $\mathcal{A}\rightarrow\widetilde{\mathcal{A}}$ is transfered in any case, since the algebras $\mathbb{C}_{n+1}$ with signatures $p-q\equiv 3,7 \pmod 8$ and $p-q\equiv 1,5 \pmod 8$ are isomorphic. In so doing, the condition (27) takes a form

$$\begin{array}{rcll}\widetilde{\omega} &=& \omega & \text{if } p-q\equiv 3,7 \pmod 8,\\ \widetilde{i\omega} &=& i\omega & \text{if } p-q\equiv 1,5 \pmod 8.\end{array}$$

Besides, each of these equalities satisfies the condition $(\varepsilon\omega)^2=1$.

Let us consider now the antiautomorphism $\mathcal{A}\rightarrow\widetilde{\mathcal{A}^\star}$. It is obvious that for the transfer of $\mathcal{A}\rightarrow\widetilde{\mathcal{A}^\star}$ from $\mathbb{C}_{n+1}$ to $\mathbb{C}_n$ it is necessary that

$$\widetilde{(\varepsilon\omega)^\star}=\varepsilon\omega. \tag{31}$$

It is easy to see that the mapping $\mathbb{C}_{p+q}\rightarrow\mathbb{C}_{p+q-1}$, where $p+q\equiv 1,5 \pmod 8$, in virtue of (28) and the second equality of (30), satisfies the condition (31), since in this case

$$\widetilde{(\varepsilon\omega)^\star}=\varepsilon\widetilde{\omega^\star}=-\varepsilon\omega^\star=\varepsilon\omega.$$

Hence it immediately follows that over the field $\mathbb{F}=\mathbb{R}$ the antiautomorphism $\mathcal{A}\rightarrow\widetilde{\mathcal{A}^\star}$ at the mappings $C\!\ell_{p,q}\rightarrow C\!\ell_{p-1,q}$, $C\!\ell_{p,q}\rightarrow C\!\ell_{p,q-1}$ $(p-q\equiv 3,7 \pmod 8)$ is not transferred.

Summarizing obtained above results we come to the following

Theorem 8. *1) If $\mathbb{F}=\mathbb{C}$ and $\mathbb{C}_{p+q}\simeq\mathbb{C}_{p+q-1}\oplus\mathbb{C}_{p+q-1}$, where $p+q\equiv 1,3,5,7 \pmod 8$, then the antiautomorphism $\mathcal{A}\rightarrow\widetilde{\mathcal{A}}$ at the homomorphic mapping $\epsilon:\mathbb{C}_{p+q}\rightarrow\mathbb{C}_{p+q-1}$ is transferred into a quotient algebra ${}^\epsilon\mathbb{C}_{p+q-1}$ in any case, the automorphism $\mathcal{A}\rightarrow\mathcal{A}^\star$ is not transferred, and the antiautomorphism $\mathcal{A}\rightarrow\widetilde{\mathcal{A}^\star}$ is transferred in the case of $p+q\equiv 1,5 \pmod 8$.*
2) If $\mathbb{F}=\mathbb{R}$ and $C\!\ell_{p,q}\simeq C\!\ell_{p-1,q}\oplus C\!\ell_{p-1,q}$, $C\!\ell_{p,q}\simeq C\!\ell_{p,q-1}\oplus C\!\ell_{p,q-1}$, where $p-q\equiv 3,7 \pmod 8$, then at the homomorphic mappings $\epsilon: C\!\ell_{p,q}\rightarrow C\!\ell_{p-1,q}$ and $\epsilon: C\!\ell_{p,q}\rightarrow C\!\ell_{p,q-1}$ the antiautomorphism $\mathcal{A}\rightarrow\widetilde{\mathcal{A}}$ is transferred correspondingly into quotient algebras ${}^\epsilon C\!\ell_{p-1,q}$ and ${}^\epsilon C\!\ell_{p,q-1}$ in any case, and the automorphism $\mathcal{A}\rightarrow\mathcal{A}^\star$ and antiautomorphism $\mathcal{A}\rightarrow\widetilde{\mathcal{A}^\star}$ are not transferred.

4 Automorphism Groups of $C\!\ell_{p,q}$, $\mathbb{C}_{p+q}$ and Discrete Transformations of $O(p,q)$, $O(p+q,\mathbb{C})$

As noted above, there exists a close relation between Dąbrowski groups $\mathbf{Pin}^{a,b,c}(p,q)$ and discrete tansformations of the orthogonal group $O(p,q)$ (in particular, Lorentz

group $O(1,3)$) [DWGK, Ch97, Ch94b, AlCh94, AlCh96]. On the other hand, discrete transformations of the group $O(p,q)$ which acting in the space $\mathbb{R}^{p,q}$ associated with the algebra $C\ell_{p,q}$, may be realized via the fundamental automorphisms of $C\ell_{p,q}$. In essence, the group $\mathbf{Pin}(p,q)$ is an intrinsic notion of $C\ell_{p,q}$, since in accordance with (11) $\mathbf{Pin}(p,q) \subset C\ell_{p,q}$. Let us show that the Dąbrowski group $\mathbf{Pin}^{a,b,c}(p,q)$ is also completely defined in the framework of the algebra $C\ell_{p,q}$, that is, there is an equivalence between $\mathbf{Pin}^{a,b,c}(p,q)$ and the group $\mathbf{Pin}(p,q) \subset C\ell_{p,q}$ complemented by the transformations $\mathcal{A} \rightarrow \mathcal{A}^\star$, $\mathcal{A} \rightarrow \widetilde{\mathcal{A}}$, $\mathcal{A} \rightarrow \widetilde{\mathcal{A}^\star}$ (in connection with this it should be noted that the Gauss-Klein group $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ is a finite group corresponded to the algebra $C\ell_{1,0} = \Omega$ [Sal81a, Sal84]).

Proposition 1. *Let $C\ell_{p,q}$ $(p+q=2m)$ be a Clifford algebra over the field $\mathbb{F}=\mathbb{R}$ and let $\mathbf{Pin}(p,q)$ be a double covering of the orthogonal group $O(p,q) = O_0(p,q) \odot \{1, P, T, PT\} \simeq O_0(p,q) \odot (\mathbb{Z}_2 \otimes \mathbb{Z}_2)$ of transformations of the space $\mathbb{R}^{p,q}$, where $\{1, P, T, PT\} \simeq \mathbb{Z}_2 \otimes \mathbb{Z}_2$ is a group of discrete transformations of $\mathbb{R}^{p,q}$, $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ is the Gauss-Klein group. Then there is an isomorphism between the group $\{1, P, T, PT\}$ and an automorphism group $\{\mathrm{Id}, \star, \widetilde{\ }, \widetilde{\star}\}$ of the algebra $C\ell_{p,q}$. In this case, parity reversal P, time reversal T and combination PT are correspond respectively to the fundamental automorphisms $\mathcal{A} \rightarrow \mathcal{A}^\star$, $\mathcal{A} \rightarrow \widetilde{\mathcal{A}}$ and $\mathcal{A} \rightarrow \widetilde{\mathcal{A}^\star}$.*

Proof. As known, the transformations $1, P, T, PT$ at the conditions $P^2 = T^2 = (PT)^2 = 1$, $PT = TP$ form an abelian group with the following multiplication table

	1	P	T	PT
1	1	P	T	PT
P	P	1	PT	T
T	T	PT	1	P
PT	PT	T	P	1

Analogously, for the automorphism group $\{\mathrm{Id}, \star, \widetilde{\ }, \widetilde{\star}\}$ in virtue of the commutativity $\widetilde{(\mathcal{A}^\star)} = (\widetilde{\mathcal{A}})^\star$ and the conditions $(\star)^2 = (\widetilde{\ })^2 = \mathrm{Id}$ a following multiplication table takes place

	Id	$\star$	$\widetilde{\ }$	$\widetilde{\star}$
Id	Id	$\star$	$\widetilde{\ }$	$\widetilde{\star}$
$\star$	$\star$	Id	$\widetilde{\star}$	$\widetilde{\ }$
$\widetilde{\ }$	$\widetilde{\ }$	$\widetilde{\star}$	Id	$\star$
$\widetilde{\star}$	$\widetilde{\star}$	$\widetilde{\ }$	$\star$	Id

The identity of the multiplication tables proves the isomorphism of the groups $\{1, P, T, PT\}$ and $\{\mathrm{Id}, \star, \widetilde{\ }, \widetilde{\star}\}$. $\square$

Further, in the case of anticommutativity $PT = -TP$ and $P^2 = T^2 = (PT)^2 = \pm 1$ an isomorphism between the group $\{1, P, T, PT\}$ and an automorphism group $\{\mathsf{I}, \mathsf{W}, \mathsf{E}, \mathsf{C}\}$, where W, E and C in accordance with (19), (17) and (21) are the matrix representations of the automorphisms $\mathcal{A} \rightarrow \mathcal{A}^\star$, $\mathcal{A} \rightarrow \widetilde{\mathcal{A}}$ and $\mathcal{A} \rightarrow \widetilde{\mathcal{A}^\star}$, is analogously proved.

Example. According to (22), (23), (24) and (25) the matrix representation of the fundamental automorphisms of the Dirac algebra $\mathbb{C}_4$ is defined by the following expressions: $\mathsf{W} = \gamma_1\gamma_2\gamma_3\gamma_4$, $\mathsf{E} = \gamma_1\gamma_3$, $\mathsf{C} = \gamma_2\gamma_4$. The multiplication table of the group $\{\mathsf{I}, \mathsf{W}, \mathsf{E}, \mathsf{C}\} \sim \{I, \gamma_1\gamma_2\gamma_3\gamma_4, \gamma_1\gamma_3, \gamma_2\gamma_4\}$ has a form

	I	$\gamma_1\gamma_2\gamma_3\gamma_4$	$\gamma_1\gamma_3$	$\gamma_2\gamma_4$
I	I	$\gamma_1\gamma_2\gamma_3\gamma_4$	$\gamma_1\gamma_3$	$\gamma_2\gamma_4$
$\gamma_1\gamma_2\gamma_3\gamma_4$	$\gamma_1\gamma_2\gamma_3\gamma_4$	I	$\gamma_2\gamma_4$	$\gamma_1\gamma_3$
$\gamma_1\gamma_3$	$\gamma_1\gamma_3$	$\gamma_2\gamma_4$	$-I$	$-\gamma_1\gamma_2\gamma_3\gamma_4$
$\gamma_2\gamma_4$	$\gamma_2\gamma_4$	$\gamma_1\gamma_3$	$-\gamma_1\gamma_2\gamma_3\gamma_4$	$-I$

$\sim$

	I	W	E	C
I	I	W	E	C
W	W	I	C	E
E	E	C	$-\mathsf{I}$	$-\mathsf{W}$
C	C	E	$-\mathsf{W}$	$-\mathsf{I}$

. (32)

However, in this representation we cannot directly to identify $\mathsf{W} = \gamma_1\gamma_2\gamma_3\gamma_4$ with the parity reversal P, since in this case the Dirac equation $(i\gamma_4\frac{\partial}{\partial x_4} - i\boldsymbol{\gamma}\frac{\partial}{\partial \mathbf{x}} - m)\psi(x_4, \mathbf{x}) = 0$ to be not invariant with respect to P. On the other hand, for the canonical basis (22) there exists a standard representation $P = \gamma_4$, $T = \gamma_1\gamma_3$ [BLP89]. The multiplication table of a group $\{1, P, T, PT\} \sim \{I, \gamma_4, \gamma_2\gamma_3, \gamma_4\gamma_1\gamma_3\}$ has a form

	I	γ_4	$\gamma_1\gamma_3$	$\gamma_4\gamma_1\gamma_3$
1	I	γ_4	$\gamma_1\gamma_3$	$\gamma_4\gamma_1\gamma_3$
γ_4	γ_4	I	$\gamma_4\gamma_1\gamma_3$	$\gamma_1\gamma_3$
$\gamma_1\gamma_3$	$\gamma_1\gamma_3$	$\gamma_4\gamma_1\gamma_3$	$-I$	$-\gamma_4$
$\gamma_4\gamma_1\gamma_3$	$\gamma_4\gamma_1\gamma_3$	$\gamma_1\gamma_3$	$-\gamma_4$	$-I$

$\sim$

	1	P	T	PT
1	1	P	T	PT
P	P	1	PT	T
T	T	PT	-1	$-P$
PT	PT	T	$-P$	-1

. (33)

It is easy to see that the tables (32) and (33) are equivalent, therefore we have an isomorphism $\{\mathsf{I}, \mathsf{W}, \mathsf{E}, \mathsf{C}\} \simeq \{1, P, T, PT\}$. Besides, each of these groups is isomorphic to the group $\mathbb{Z}_4$.

Theorem 9. *Let* $\mathsf{A} = \{\mathsf{I},\ \mathsf{W},\ \mathsf{E},\ \mathsf{C}\}$ *be the automorphism group of the algebras* $C\!\ell_{p,q}$, $\mathbb{C}_{p+q}$ $(p+q=2m)$, *where* $\mathsf{W} = \mathcal{E}_1\mathcal{E}_2\cdots\mathcal{E}_m\mathcal{E}_{m+1}\mathcal{E}_{m+2}\cdots\mathcal{E}_{p+q}$, *and* $\mathsf{E} = \mathcal{E}_1\mathcal{E}_2\cdots\mathcal{E}_m$, $\mathsf{C} = \mathcal{E}_{m+1}\mathcal{E}_{m+2}\cdots\mathcal{E}_{p+q}$ *if* $m \equiv 1 \pmod 2$, *and* $\mathsf{E} = \mathcal{E}_{m+1}\mathcal{E}_{m+2}\cdots\mathcal{E}_{p+q}$, $\mathsf{C} = \mathcal{E}_1\mathcal{E}_2\cdots\mathcal{E}_m$ *if* $m \equiv 0 \pmod 2$. *Let* A_- *and* A_+ *be the automorphism groups, in which the all elements respectively commute* ($m \equiv 0 \pmod 2$) *and anticommute* ($m \equiv 1 \pmod 2$). *Then there are the following isomorphisms between finite groups and automorphism groups with different signatures* $(a,\ b,\ c)$, *where* $a, b, c \in \{-, +\}$:
1) $\mathbb{F} = \mathbb{R}$. $\mathbf{A}_- \simeq \mathbb{Z}_2 \otimes \mathbb{Z}_2$ *for the signature* $(+,\ +,\ +)$ *if* $p-q \equiv 0, 4 \pmod 8$. $\mathbf{A}_- \simeq \mathbb{Z}_4$ *for* $(+,\ -,\ -)$ *if* $p-q \equiv 0, 4 \pmod 8$ *and for* $(-,\ +,\ -)$, $(-,\ -,\ +)$ *if* $p-q \equiv 2, 6 \pmod 8$. $\mathbf{A}_+ \simeq Q_4/\mathbb{Z}_2$ *for* $(-,\ -,\ -)$ *if* $p-q \equiv 2, 6 \pmod 8$. $\mathbf{A}_+ \simeq D_4/\mathbb{Z}_2$ *for* $(-,\ +,\ +)$ *if* $p-q \equiv 2, 6 \pmod 8$ *and for* $(+,\ -,\ +)$, $(+,\ +,\ -)$ *if* $p-q \equiv 0, 4 \pmod 8$.
2) Over the field $\mathbb{F} = \mathbb{C}$ *there are only two non-isomorphic groups:* $\mathbf{A}_- \simeq \mathbb{Z}_2 \otimes \mathbb{Z}_2$ *for the signature* $(+,\ +,\ +)$ *if* $p-q \equiv 0, 4 \pmod 8$ *and* $\mathbf{A}_+ \simeq Q_4/\mathbb{Z}_2$ *for* $(-,\ -,\ -)$ *if* $p-q \equiv 2, 6 \pmod 8$.

Proof. First of all, since $\omega^2 = +1$ if $p-q \equiv 0, 4 \pmod 8$ and $\omega^2 = -1$ if $p-q \equiv 2, 6 \pmod 8$, then in the case of $\mathbb{F} = \mathbb{R}$ for the matrix of the automorphism $\mathcal{A} \to \mathcal{A}^\star$ we have

$$\mathsf{W} = \begin{cases} +\mathsf{I}, & \text{if } p-q \equiv 0, 4 \pmod 8; \\ -\mathsf{I}, & \text{if } p-q \equiv 2, 6 \pmod 8. \end{cases}$$

Over the field $\mathbb{F} = \mathbb{C}$ we can always to suppose $\mathsf{W}^2 = 1$. Further, let us find now the matrix E of the antiautomorphism $\mathcal{A} \to \widetilde{\mathcal{A}}$ at any $n = 2m$, and elucidate the conditions at which the matrix E commutes with W, and also define a square of the matrix E. Follows to [Ras55] let introduce along with the algebra $\mathbb{C}_{p+q}$ an auxiliary algebra $\mathbb{C}_m$ with basis elements

$$1,\ \varepsilon_\alpha,\ \varepsilon_{\alpha_1\alpha_2}\ (\alpha_1 < \alpha_2),\ \varepsilon_{\alpha_1\alpha_2\alpha_3}\ (\alpha_1 < \alpha_2 < \alpha_3),\ \ldots\ \varepsilon_{12\ldots m}.$$

In so doing, linear operators $\hat{\mathcal{E}}_i$ acting in the space $\mathbb{C}^m$ associated with the algebra $\mathbb{C}_m$, are defined by a following rule

$$\begin{aligned} \hat{\mathcal{E}}_j\quad &:\ \Lambda \longrightarrow \Lambda\varepsilon_j, \\ \hat{\mathcal{E}}_{m+j}\ &:\ \Lambda^1 \longrightarrow -i\varepsilon_j\Lambda^1,\ \Lambda^0 \longrightarrow i\varepsilon_j\Lambda^0, \end{aligned} \tag{34}$$

where Λ is a general element of the auxiliary algebra $\mathbb{C}_m$, Λ^1 and Λ^0 are correspondingly odd and even parts of Λ, ε_j are units of the auxiliary algebra,

$j = 1, 2, \ldots m$. Analogously, in the case of matrix representations of $C\!\ell_{p,q}$ we have

$$\begin{aligned}\hat{\mathcal{E}}_j \quad &: \Lambda \longrightarrow \Lambda\beta_j\varepsilon_j,\\ \hat{\mathcal{E}}_{m+j} &: \Lambda^1 \longrightarrow -\varepsilon_j\beta_{m+j}\Lambda^1,\ \Lambda^0 \longrightarrow \varepsilon_j\beta_{m+j}\Lambda^0,\end{aligned}\tag{35}$$

where β_i are arbitrary complex numbers. It is easy to verify that transposition of the matrices of so defined operators gives

$$\mathcal{E}_j^T = \mathcal{E}_j, \quad \mathcal{E}_{m+j}^T = -\mathcal{E}_{m+j}.\tag{36}$$

Further, for the antiautomorphism $\mathcal{A} \rightarrow \widetilde{\mathcal{A}}:\ \widetilde{\mathsf{A}} = \mathsf{E}\mathsf{A}^T\mathsf{E}^{-1}$, since in this case $\mathbf{e}_i \rightarrow \mathbf{e}_i$, it is sufficient to select the matrix E so that

$$\mathsf{E}\mathcal{E}_i^T\mathsf{E}^{-1} = \mathcal{E}_i$$

or taking into account (36)

$$\mathsf{E}\mathcal{E}_j^T\mathsf{E}^{-1} = \mathcal{E}_j, \quad \mathsf{E}\mathcal{E}_{m+j}^T\mathsf{E}^{-1} = -\mathcal{E}_{m+j}.\tag{37}$$

Therefore, if m is odd, then the matrix E has a form

$$\mathsf{E} = \mathcal{E}_1\mathcal{E}_2\ldots\mathcal{E}_m,\tag{38}$$

since in this case a product $\mathcal{E}_1\mathcal{E}_2\ldots\mathcal{E}_m$ commutes with all elements $\mathcal{E}_j$ ($j = 1, \ldots m$) and anticommutes with all elements $\mathcal{E}_{m+j}$. Analogously, if m is even, then

$$\mathsf{E} = \mathcal{E}_{m+1}\mathcal{E}_{m+2}\ldots\mathcal{E}_{p+q}.\tag{39}$$

As required according to (37) in this case a product (39) commutes with $\mathcal{E}_j$ and anticommutes with $\mathcal{E}_{m+j}$.

Let us consider now the conditions at which the matrix E commutes or anticommutes with W. Let $\mathsf{E} = \mathcal{E}_1\mathcal{E}_2\ldots\mathcal{E}_m$, where m is odd, since $\mathsf{W} = \mathcal{E}_1\ldots\mathcal{E}_m\mathcal{E}_{m+1}\ldots\mathcal{E}_{p+q}$, then

$$\begin{aligned}\mathcal{E}_1\ldots\mathcal{E}_m\mathcal{E}_1\ldots\mathcal{E}_m\mathcal{E}_{m+1}\ldots\mathcal{E}_{p+q} &= (-1)^{\frac{m(m-1)}{2}}\sigma_1\sigma_2\ldots\sigma_m\mathcal{E}_{m+1}\ldots\mathcal{E}_{p+q},\\ \mathcal{E}_1\ldots\mathcal{E}_m\mathcal{E}_{m+1}\ldots\mathcal{E}_{p+q}\mathcal{E}_1\ldots\mathcal{E}_m &= (-1)^{\frac{m(3m-1)}{2}}\sigma_1\sigma_2\ldots\sigma_m\mathcal{E}_{m+1}\ldots\mathcal{E}_{p+q},\end{aligned}$$

where σ_i are the functions of the form (2). It is easy to see that in this case the elements W and E are always anticommute. Indeed, a comparison $\frac{m(3m-1)}{2} \equiv \frac{m(m-1)}{2} \pmod 2$ is equivalent to $m^2 \equiv 0, 1 \pmod 2$, and since m is odd, then we have always $m^2 \equiv 1 \pmod 2$. At m is even and $\mathsf{E} = \mathcal{E}_{m+1}\ldots\mathcal{E}_{p+q}$ it is easy to see that the matrices W and E are always commute ($m \equiv 0 \pmod 2$). Further, let r be a quantity of the elements $\mathcal{E}_j$ of the product (38) whose squares equal

to $+\mathsf{I}$, and let s be a quantity of the elements $\mathcal{E}_{m+j}$ of the product (38) whose squares equal to $-\mathsf{I}$. Then a square of the matrix (38) at m is odd equals to $+\mathsf{I}$ if $r-s\equiv 3 \pmod 4$ and respectively $-\mathsf{I}$ if $r-s\equiv 1 \pmod 4$. Analogously, a square of the matrix (39) at m is even equals to $+\mathsf{I}$ if $k-t\equiv 0 \pmod 4$ and respectively $-\mathsf{I}$ if $k-t\equiv 2 \pmod 4$. It is obvious that over the field $\mathbb{F}=\mathbb{C}$ we can to suppose $\mathcal{E}^2_{1\ldots m}=\mathcal{E}^2_{m+1\ldots p+q}=\mathsf{I}$.

Let us find now the matrix C of the antiautomorphism $\mathcal{A}\rightarrow\widetilde{\mathcal{A}^\star}$: $\widetilde{\mathsf{A}^\star}=\mathsf{C}\mathsf{A}^T\mathsf{C}^{-1}$. Since in this case $\mathbf{e}_i\rightarrow -\mathbf{e}_i$, then it is sufficient to select the matrix C so that

$$\mathsf{C}\mathcal{E}^T_i\mathsf{C}^{-1}=-\mathcal{E}_i$$

or taking into account (36)

$$\mathsf{C}\mathcal{E}^T_j\mathsf{C}^{-1}=-\mathcal{E}_j,\quad \mathsf{C}\mathcal{E}^T_{m+j}\mathsf{C}^{-1}=\mathcal{E}_{m+j}. \tag{40}$$

where $j=1,\ldots m$. In comparison with (37) it is easy to see that in (40) the matrices $\mathcal{E}_j$ and $\mathcal{E}_{m+j}$ are changed by the roles. Therefore, if m is odd, then

$$\mathsf{C}=\mathcal{E}_{m+1}\mathcal{E}_{m+2}\ldots\mathcal{E}_{p+q}, \tag{41}$$

and if m is even, then

$$\mathsf{C}=\mathcal{E}_1\mathcal{E}_2\ldots\mathcal{E}_m. \tag{42}$$

Permutation conditions of the matrices C and W are analogous to the permutation conditions of E with W, that is, the matrix C of the form (41) always anticommutes with W ($m\equiv 1 \pmod 2$), and the matrix C of the form (42) always commutes with W ($m\equiv 0 \pmod 2$). Correspondingly, a square of the matrix (41) equals to $+\mathsf{I}$ if $k-t\equiv 3 \pmod 4$ and $-\mathsf{I}$ if $k-t\equiv 1 \pmod 4$. Analogously, a square of the matrix (42) equals to $+\mathsf{I}$ if $r-s\equiv 0 \pmod 4$ and $-\mathsf{I}$ if $r-s\equiv 2 \pmod 4$. Obviously, over the field $\mathbb{F}=\mathbb{C}$ we can suppose $\mathsf{C}^2=\mathsf{I}$.

Finally, let us find permutation conditions of the matrices E and C. First of all, at m is odd $\mathsf{E}=\mathcal{E}_1\ldots\mathcal{E}_m$, $\mathsf{C}=\mathcal{E}_{m+1}\ldots\mathcal{E}_{p+q}$, alternatively, at m is even $\mathsf{E}=\mathcal{E}_{m+1}\ldots\mathcal{E}_{p+q}$, $\mathsf{C}=\mathcal{E}_1\ldots\mathcal{E}_m$. Therefore,

$$\mathcal{E}_1\ldots\mathcal{E}_m\mathcal{E}_{m+1}\ldots\mathcal{E}_{p+q}=(-1)^{m^2}\mathcal{E}_{m+1}\ldots\mathcal{E}_{p+q}\mathcal{E}_1\ldots\mathcal{E}_m,$$

that is, the matrices E and C commute at $m\equiv 0 \pmod 2$ and anticommute at $m\equiv 1 \pmod 2$.

Now we have all the necessary conditions for the definition and classification of isomorphisms between finite groups and automorphism groups of Clifford algebras. Let $\mathbb{F}=\mathbb{R}$ and let $m\equiv 0 \pmod 2$, therefore, a group $\mathsf{A}=\{\mathsf{I},\,\mathsf{W},\,\mathsf{E},\,\mathsf{C}\}$ is Abelian. A condition $\mathsf{W}^2=\mathsf{E}^2=\mathsf{C}^2=\mathsf{I}$ is equivalent to $p-q\equiv 0,4 \pmod 8$, $r-s\equiv 0 \pmod 4$, $k-t\equiv 0 \pmod 4$, which, clearly, are compatible. In accordance with (38)–(40) and (41)–(42) at $m\equiv 0 \pmod 2$ $\mathsf{W}=\mathsf{C}\mathsf{E}$ and

$\mathsf{W}^2 = (\mathsf{CE})^2 = \mathsf{CECE} = \mathsf{CCEE} = \mathsf{I}$. Therefore, $\mathsf{A}_- \simeq \mathbb{Z}_2 \otimes \mathbb{Z}_2$ for the signature $(+,+,+)$ if $p - q \equiv 0, 4 \pmod 8$. The isomorphism $\mathsf{A}_- \simeq \mathbb{Z}_4$ for the signatures $(+,-,-)$ ($p - q \equiv 0, 4 \pmod 8$) and $(-,+,-)$, $(-,-,+)$ ($p - q \equiv 2, 6 \pmod 8$) is analogously proved. It is easy to see that for $m \equiv 0 \pmod 2$ there are only four isomorphisms considered previously. Further, for $m \equiv 1 \pmod 2$ all the elements of the group A anticommute and in this case $\mathsf{W} = \mathsf{EC}$. The signature $(-,-,-)$ is equivalent to conditions $p - q \equiv 2, 6 \pmod 8$, $r - s \equiv 1 \pmod 4$, $k - t \equiv 1 \pmod 4$, here $\mathsf{W}^2 = (\mathsf{EC})^2 = \mathsf{ECEC} = -\mathsf{EECC} = -\mathsf{I}$ and we have an isomorphism $\mathsf{A}_+ \simeq Q_4/\mathbb{Z}_2$, where $Q_4/\mathbb{Z}_2 = \{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$, $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are the quaternion units. It is easy to verify that for the signatures $(-,+,+)$ at $p - q \equiv 2, 6 \pmod 8$ and $(+,-,+)$, $(+,+,-)$ at $p - q \equiv 0, 4 \pmod 8$ we have an isomorphism $\mathsf{A}_+ \simeq D_4/\mathbb{Z}_2$, where $D_4/\mathbb{Z}_2 = \{1, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_{12}\}$, $\mathbf{e}_1, \mathbf{e}_2$ are the units of the algebra $C\!\ell_{1,1}$ or $C\!\ell_{2,0}$. The eight automorphism groups considered previously, each of which is isomorphic to one from the four finite groups $\mathbb{Z}_2 \otimes \mathbb{Z}_2$, $\mathbb{Z}_4$, $Q_4/\mathbb{Z}_2$, $D_4/\mathbb{Z}_2$, are the only possible over the field $\mathbb{F} = \mathbb{R}$. In contrast with this, over the field $\mathbb{F} = \mathbb{C}$ we can suppose $\mathsf{W}^2 = \mathsf{E}^2 = \mathsf{C}^2 = \mathsf{I}$. At $m \equiv 0 \pmod 2$ we have only one signature $(+,+,+)$ and an isomorphism $\mathsf{A}_- \simeq \mathbb{Z}_2 \otimes \mathbb{Z}_2$ if $p + q \equiv 0, 4 \pmod 8$, since over the field $\mathbb{C}$ the signatures $(+,-,-)$, $(-,+,-)$ and $(-,--)$ are isomorphic to $(+,+,+)$. Correspondingly, at $m \equiv 1 \pmod 2$ we have an isomorphism $\mathsf{A}_+ \simeq Q_4/\mathbb{Z}_2$ for the signature $(-,-,-)$ if $p + q \equiv 2, 6 \pmod 8$. It should be noted that the signatures $(+,+,+)$ and $(-,-,-)$ are non-isomorphic, since there exists no a group A with the signature $(+,+,+)$ in which all the elements anticommute, and also there exists no a group A with $(-,-,-)$ in which the all elements commute. Thus, over the field $\mathbb{C}$ we have only two non-isomorphic automorphism groups: $\mathsf{A}_- \simeq \mathbb{Z}_2 \otimes \mathbb{Z}_2$, $\mathsf{A}_+ \simeq Q_4/\mathbb{Z}_2$. $\square$

The following Theorem is a direct consequence of the previous Theorem. Here we establish a relation between signatures (a, b, c) of the Dąbrowski groups and signatures (p, q) of the Clifford algebras with even dimensionality.

Theorem 10. *Let $\mathbf{Pin}^{a,b,c}(p, q)$ be a double covering of the orthogonal group $O(p, q)$ of the space $\mathbb{R}^{p,q}$ associated with the algebra $C\!\ell_{p,q}$ and let $\mathbf{Pin}^{a,b,c}(p+q, \mathbb{C})$ be a double covering of the complex orthogonal group $O(p+q, \mathbb{C})$ of the space $\mathbb{C}^{p+q}$ associated with the algebra $\mathbb{C}_{p+q}$. Dimensionalities of the algebras $C\!\ell_{p,q}$ and $\mathbb{C}_{p+q}$ are even ($p + q = 2m$), squares of the symbols $a, b, c \in \{-,+\}$ are correspond to squares of the elements of the finite group $\mathsf{A} = \{\mathsf{I}, \mathsf{W}, \mathsf{E}, \mathsf{C}\}$: $a = \mathsf{W}^2$, $b = \mathsf{E}^2$, $c = \mathsf{C}^2$, where W, E and C are correspondingly the matrices of the fundamental automorphisms $\mathcal{A} \rightarrow \mathcal{A}^\star$, $\mathcal{A} \rightarrow \widetilde{\mathcal{A}}$ and $\mathcal{A} \rightarrow \widetilde{\mathcal{A}^\star}$ of $C\!\ell_{p,q}$ and $\mathbb{C}_{p+q}$. Then over the field $\mathbb{K} = \mathbb{R}$ for the algebra $C\!\ell_{p,q}$ there are eight double coverings of the group $O(p, q)$ and two non-isomorphic double coverings of the group $O(p+q, \mathbb{C})$ for $\mathbb{C}_{p+q}$ over the field $\mathbb{K} = \mathbb{C}$:*
1) $\mathbb{F} = \mathbb{R}$. Non–Cliffordian groups

$$\mathbf{Pin}^{+,+,+}(p, q) \simeq \frac{(\mathbf{Spin}_0(p, q) \odot \mathbb{Z}_2 \otimes \mathbb{Z}_2 \otimes \mathbb{Z}_2)}{\mathbb{Z}_2},$$

if $p-q\equiv 0,4 \pmod 8$ *and*

$$\mathbf{Pin}^{a,b,c}(p,q)\simeq\frac{(\mathbf{Spin}_0(p,q)\odot(\mathbb{Z}_2\otimes\mathbb{Z}_4)}{\mathbb{Z}_2},$$

if $(a,b,c)=(+,-,-)$ *and* $p-q\equiv 0,4 \pmod 8$, *and also* $(a,b,c)=\{(-,+,-),\,(-,-,+)\}$ *if* $p-q\equiv 2,6 \pmod 8$.

Cliffordian groups

$$\mathbf{Pin}^{-,-,-}(p,q)\simeq\frac{(\mathbf{Spin}_0(p,q)\odot Q_4)}{\mathbb{Z}_2},$$

if $p-q\equiv 2,6 \pmod 8$ *and*

$$\mathbf{Pin}^{a,b,c}(p,q)\simeq\frac{(\mathbf{Spin}_0(p,q)\odot D_4)}{\mathbb{Z}_2},$$

if $(a,b,c)=(-,+,+)$ *and* $p-q\equiv 2,6 \pmod 8$, *and also if* $(a,b,c)=\{(+,-,+),\,(+,+,-)\}$ *and* $p-q\equiv 0,4 \pmod 8$.

2) $\mathbb{F}=\mathbb{C}$. *A non-Cliffordian group*

$$\mathbf{Pin}^{+,+,+}(p+q,\mathbb{C})\simeq\frac{(\mathbf{Spin}_0(p+q,\mathbb{C})\odot\mathbb{Z}_2\otimes\mathbb{Z}_2\otimes\mathbb{Z}_2)}{\mathbb{Z}_2},$$

if $p+q\equiv 0,4 \pmod 8$. *A Cliffordian group*

$$\mathbf{Pin}^{-,-,-}(p+q,\mathbb{C})\simeq\frac{(\mathbf{Spin}_0(p+q,\mathbb{C})\odot Q_4)}{\mathbb{Z}_2},$$

if $p+q\equiv 2,6 \pmod 8$.

5 Dąbrowski Groups for Odd-dimensional Spaces

According to Theorem 8 and Theorem 4 in the case of odd-dimensional spaces $\mathbb{R}^{p,q}$ and $\mathbb{C}^{p+q}$ the algebra homomorphisms $C\!\ell_{p,q}\to C\!\ell_{p-1,q}$, $C\!\ell_{p,q}\to C\!\ell_{p,q-1}$ and $\mathbb{C}_{p+q}\to\mathbb{C}_{p+q-1}$ induce group homomorphisms $\mathbf{Pin}(p,q)\to\mathbf{Pin}(p-1,q)$, $\mathbf{Pin}(p,q)\to\mathbf{Pin}(p,q-1)$, $\mathbf{Pin}(p+q,\mathbb{C})\to\mathbf{Pin}(p+q-1,\mathbb{C})$ and correspondingly $\mathbf{Pin}(p,q)\to\mathbf{Spin}(p,q)$, $\mathbf{Pin}(p+q,\mathbb{C})\to\mathbf{Spin}(p+q,\mathbb{C})$.

Theorem 11. *1) If* $\mathbb{F}=\mathbb{R}$ *and* $\mathbf{Pin}^{a,b,c}(p,q)\simeq\mathbf{Pin}^{a,b,c}(p-1,q)\cup\omega\,\mathbf{Pin}^{a,b,c}(p-1,q)$, $\mathbf{Pin}^{a,b,c}(p,q)\simeq\mathbf{Pin}^{a,b,c}(p,q-1)\cup\omega\,\mathbf{Pin}^{a,b,c}(p,q-1)$ *are the Dąbrowski groups over* $\mathbb{R}$, *where* $p-q\equiv 3,7 \pmod 8$, *then in the result of homomorphic*

mappings $\mathbf{Pin}^{a,b,c}(p,q)\rightarrow\mathbf{Pin}^{a,b,c}(p-1,q)$ *and* $\mathbf{Pin}^{a,b,c}(p,q)\rightarrow\mathbf{Pin}^{a,b,c}(p,q-1)$ *take place following quotient groups:*

$$
\begin{aligned}
\mathbf{Pin}^{b}(p-1,q) &\simeq \frac{(\mathbf{Spin}_0(p-1,q)\odot\mathbb{Z}_2\otimes\mathbb{Z}_2)}{\mathbb{Z}_2},\\
\mathbf{Pin}^{b}(p,q-1) &\simeq \frac{(\mathbf{Spin}_0(p,q-1)\odot\mathbb{Z}_2\otimes\mathbb{Z}_2)}{\mathbb{Z}_2}.
\end{aligned}
$$

2) If $\mathbb{F}=\mathbb{C}$ *and* $\mathbf{Pin}^{a,b,c}(p+q,\mathbb{C})\simeq\mathbf{Pin}^{a,b,c}(p+q-1,\mathbb{C})\cup\mathbf{Pin}^{a,b,c}(p+q-1,\mathbb{C})$ *are the Dąbrowski groups over* $\mathbb{C}$*, where* $p+q\equiv 1,3,5,7\pmod 8$*, then in the result of an homomorpic mapping* $\mathbf{Pin}^{a,b,c}(p+q,\mathbb{C})\rightarrow\mathbf{Pin}^{a,b,c}(p+q-1,\mathbb{C})$ *take place following quotient groups:*

$$
\mathbf{Pin}^{b}(p+q-1,\mathbb{C})\simeq\frac{(\mathbf{Spin}_0(p+q-1,\mathbb{C})\odot\mathbb{Z}_2\otimes\mathbb{Z}_2)}{\mathbb{Z}_2},
$$

if $p+q\equiv 3,7\pmod 8$ *and*

$$
\mathbf{Pin}^{b,c}(p+q-1,\mathbb{C}),
$$

if $p+q\equiv 1,5\pmod 8$*, at this point a set of the fundamental automorphisms, which correspond to the discrete transformations of the space* $\mathbb{C}^{p+q-1}$ *associated with a quotient algebra* ${}^{\epsilon}\mathbb{C}_{p+q-1}$*, does not form a finite group.*

Proof. Indeed, over the field $\mathbb{F}=\mathbb{R}$ in accordance with Theorem 8 from all the fundamental automorphisms at the homomorphic mappings $C\!\ell_{p,q}\rightarrow C\!\ell_{p-1,q}$ and $C\!\ell_{p,q}\rightarrow C\!\ell_{p,q-1}$ only the antiautomorphism $\mathcal{A}\rightarrow\widetilde{\mathcal{A}}$ is transferred into quotient algebras ${}^{\epsilon}C\!\ell_{p-1,q}$ and ${}^{\epsilon}C\!\ell_{p,q-1}$. Further, according to Proposition 1 the antiautomorphism $\mathcal{A}\rightarrow\widetilde{\mathcal{A}}$ corresponds to time reversal T. Therefore, groups of the discrete transformations of the spaces $\mathbb{R}^{p-1,q}$ and $\mathbb{R}^{p,q-1}$ associated with the quotient algebras ${}^{\epsilon}C\!\ell_{p-1,q}$ and ${}^{\epsilon}C\!\ell_{p,q-1}$ are defined by a two–element group $\{1,T\}\sim\{\mathsf{I},\mathsf{E}\}\simeq\mathbb{Z}_2$, where $\{\mathsf{I},\mathsf{E}\}$ is an automorphism group of the quotient algebras ${}^{\epsilon}C\!\ell_{p-1,q}$, ${}^{\epsilon}C\!\ell_{p-1,q}$, E is a matrix of the antiautomorphism $\mathcal{A}\rightarrow\widetilde{\mathcal{A}}$. Thus, at $p-q\equiv 3,7\pmod 8$ there are the homomorphic mappings $\mathbf{Pin}^{a,b,c}(p,q)\rightarrow\mathbf{Pin}^{b}(p-1,q)$ and $\mathbf{Pin}^{a,b,c}(p,q)\rightarrow\mathbf{Pin}^{b}(p,q-1)$, where $\mathbf{Pin}^{b}(p-1,q)$, $\mathbf{Pin}^{b}(p,q-1)$ are quotient groups, $b=T^2=\mathsf{E}^2$. At this point, a double covering of C^b is isomorphic to $\mathbb{Z}_2\otimes\mathbb{Z}_2$.

Analogously, over the field $\mathbb{F}=\mathbb{C}$ at $p+q\equiv 3,7\pmod 8$ we have a quotient group $\mathbf{Pin}^{b}(p+q-1,\mathbb{C})$. Further, according to Theorem 8 in the result of the homomorphic mapping $\mathbb{C}_{p+q}\rightarrow\mathbb{C}_{p+q-1}$ the antiautomorphisms $\mathcal{A}\rightarrow\widetilde{\mathcal{A}}$, $\mathcal{A}\rightarrow\widetilde{\mathcal{A}^{\star}}$ are transferred into a quotient algebra ${}^{\epsilon}\mathbb{C}_{p+q-1}$ at $p+q\equiv 1,5\pmod 8$. Therefore, a set of the discrete transformations of the space $\mathbb{C}^{p+q-1}$ associated with the quotient algebra ${}^{\epsilon}\mathbb{C}_{p+q-1}$ is defined by a three-element set $\{1,T,PT\}\sim\{\mathsf{I},\mathsf{E},\mathsf{C}\}$, where $\{\mathsf{I},\mathsf{E},\mathsf{C}\}$ is a set of the automorphisms of ${}^{\epsilon}\mathbb{C}_{p+q-1}$, E and C are correspondingly the matrices of the antiautomorphisms $\mathcal{A}\rightarrow\widetilde{\mathcal{A}}$ and $\mathcal{A}\rightarrow\widetilde{\mathcal{A}^{\star}}$. It is easy to see

that the set $\{1, T, PT\} \sim \{\mathsf{I}, \mathsf{E}, \mathsf{C}\}$ does not form a finite group. Thus, at $p+q \equiv 1, 5 \pmod 8$ there is an homomorphism $\mathbf{Pin}^{a,b,c}(p+q, \mathbb{C}) \rightarrow \mathbf{Pin}^{b,c}(p+q-1, \mathbb{C})$, where $\mathbf{Pin}^{b,c}(p+q-1, \mathbb{C})$ is a quotient group, $b = T^2 = \mathsf{E}^2$, $c = (PT)^2 = \mathsf{C}^2$. $\square$

Example. Let us consider a simplest complex Clifford algebra with odd dimensionality, $\mathbb{C}_3$. The algebra $\mathbb{C}_3$ may be represented by two different complexifications: $\mathbb{C}_3 = \mathbb{C} \otimes C\!\ell_{3,0}$ and $\mathbb{C}_3 = \mathbb{C} \otimes C\!\ell_{0,3}$, where $C\!\ell_{3,0}$ and $C\!\ell_{0,3}$ are correspondingly the algebras of hyperbolic and elliptic biquaternions. In accordance with Theorem 2 there is a decomposition of $\mathbb{C}_3$ into a direct sum of two subalgebras, which may be represented by a following scheme:

$$
\begin{tikzcd}
 & \mathbb{C}_3 \arrow[dl, "\lambda_-"'] \arrow[dr, "\lambda_+"] & \\
\mathbb{C}_2 & \oplus & \mathbb{C}_2
\end{tikzcd}
$$

Here the idempotents

$$
\lambda_- = \frac{1 - i\omega}{2}, \quad \lambda_+ = \frac{1 + i\omega}{2}
$$

in accordance with [CF96] may be identified with helicity projection operators. Further, according to Theorem 8 at the homomorphic mapping $\epsilon : \mathbb{C}_3 \rightarrow \mathbb{C}_2$ the antiautomorphisms $\mathcal{A} \rightarrow \widetilde{\mathcal{A}}$ and $\mathcal{A} \rightarrow \widetilde{\mathcal{A}^\star}$ are transferred into a quotient algebra ${}^\epsilon\mathbb{C}_2$, and the automorphism $\mathcal{A} \rightarrow \mathcal{A}^\star$ is not transferred. Therefore, there is an homomorphism $\mathbf{Pin}^{a,b,c}(3, \mathbb{C}) \rightarrow \mathbf{Pin}^{b,c}(2, \mathbb{C})$ (Theorem 11), where $\mathbf{Pin}^{b,c}(2, \mathbb{C})$ is a quotient group which double covers the orthogonal group $O(2, \mathbb{C})$ of the space $\mathbb{C}^2$ associated with ${}^\epsilon\mathbb{C}_2$. According to proposition 1 the automorphism $\mathcal{A} \rightarrow \mathcal{A}^\star$ corresponds to parity reversal P which under action of the homomorphism ϵ is not transferred into the quotient algebra ${}^\epsilon\mathbb{C}_2$ and correspondingly quotient group $\mathbf{Pin}^{b,c}(2, \mathbb{C})$. Thus, we have a 'symmetry breaking' of the group of discrete transformations with excluded operation P. In physics there is an analog of this situation known as a *parity violation*. In connection with this it pays to relate the quotient algebra ${}^\epsilon\mathbb{C}_2$ and quotient group $\mathbf{Pin}^{b,c}(2, \mathbb{C})$ with some chiral field, for example, neitrino field. As known, in Nature there exist only left neitrino and right antineitrino and there exist no right neitrino and left antineitrino, therefore, for the neitrino field the operation P is violated. In order to proceed this analogy, at first we must to establish a relation between the quotient algebra ${}^\epsilon\mathbb{C}_2$ and some spinor field. Let us show that such a field is a Dirac-Hestenes spinor field [Hest66, Hest90]. Indeed, in accordance with Theorem 1 we have $\mathbb{C}_2 \simeq C\!\ell_{3,0}$, further $C\!\ell_{3,0} \simeq C\!\ell^+_{1,3}$, where $C\!\ell_{1,3}$ is a spacetime algebra. Units of the algebra

$\mathcal{C}\ell_{1,3}$ in the matrix representation have the form

$$\Gamma_0=\begin{pmatrix} I & 0\\ 0 & -I\end{pmatrix},\;\;\Gamma_1=\begin{pmatrix} 0 & \sigma_1\\ -\sigma_1 & 0\end{pmatrix},\;\;\Gamma_2=\begin{pmatrix} 0 & \sigma_2\\ -\sigma_2 & 0\end{pmatrix},\;\;\Gamma_3=\begin{pmatrix} 0 & \sigma_3\\ -\sigma_3 & 0\end{pmatrix}, \tag{43}$$

where σ_i are the Pauli matrices

$$\sigma_1=\begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix},\quad \sigma_2=\begin{pmatrix} 0 & -i\\ i & 0\end{pmatrix},\quad \sigma_3=\begin{pmatrix} 1 & 0\\ 0 & -1\end{pmatrix},$$

I is the unit matrix. The Dirac-Hestenes spinor ϕ is an element of the algebra $\mathcal{C}\ell^+_{1,3}\simeq\mathcal{C}\ell_{3,0}$ and, therefore, may be represented by the biquaternion number

$$\phi=a^0+a^{01}\Gamma_{01}+a^{02}\Gamma_{02}+a^{03}\Gamma_{03}+a^{12}\Gamma_{12}+a^{13}\Gamma_{13}+a^{23}\Gamma_{23}+a^{0123}\Gamma_{0123}. \tag{44}$$

Or in the matrix form

$$\phi=\begin{pmatrix} \phi_1 & -\phi_2^* & \phi_3 & \phi_4^*\\ \phi_2 & \phi_1^* & \phi_4 & -\phi_3^*\\ \phi_3 & \phi_4^* & \phi_1 & -\phi_2^*\\ \phi_4 & -\phi_3^* & \phi_2 & \phi_1^*\end{pmatrix}, \tag{45}$$

where

$$\begin{aligned} \phi_1 &= a^0-ia^{12},\\ \phi_2 &= a^{13}-ia^{23},\\ \phi_3 &= a^{03}-ia^{0123},\\ \phi_4 &= a^{01}+ia^{02}. \end{aligned}$$

The spinor ϕ, defined by the expression (44) or (45), satisfies to a Dirac-Hestenes equation [Hest90]

$$\partial\phi\Gamma_{21}=\frac{mc}{\hbar}\phi\Gamma_0, \tag{46}$$

where $\partial=\Gamma^\nu\frac{\partial}{\partial x^\nu}$. Further, let $^\epsilon\phi\in{}^\epsilon\mathbb{C}_2$ be a Dirac-Hestenes 'quotient spinor'. In virtue of the isomorphism $^\epsilon\mathbb{C}_2\simeq\mathcal{C}\ell_{3,0}\simeq\mathcal{C}\ell^+_{1,3}$ we have for the spinor $^\epsilon\phi$ the representation (44). It is known [RSVL] that a transformation group of the Dirac-Hestenes field is a group $\mathbf{Spin}_+(1,3)$ which double covers the connected component of the Lorentz group, since

$$\mathbf{Spin}_+(1,3)\simeq\left\{\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in\mathbb{C}_2:\ \det\begin{pmatrix} a & b\\ c & d\end{pmatrix}=1\right\}=SL(2,\mathbb{C}).$$

It is obvious that $\mathbf{Pin}(2,\mathbb{C})\simeq\mathbf{Spin}(3,\mathbb{C})$ and further $\mathbf{Spin}(3,\mathbb{C})\simeq\mathbf{Spin}(1,3)$, whence $\mathbf{Spin}_0(3,\mathbb{C})\simeq\mathbf{Spin}_+(1,3)$, therefore, a transformation group of the quotient spinor $^\epsilon\phi$ is also isomorphic to $\mathbf{Spin}_+(1,3)$. The complete transformation group of $^\epsilon\phi$ is isomorphic to the quotient group $\mathbf{Pin}^{b,c}(2,\mathbb{C})$ consisting of

$\mathbf{Spin}_+(1,3)$ and the two discrete transformations T and PT (the latter transformation in virtue of CPT-theorem is equivalent to a charge conjugation C). Since the quotient algebra ${}^\epsilon\mathbb{C}_2$ is isomorphic to a subalgebra of all the even elements of the spacetime algebra $C\!\ell_{1,3}$, then we can express the automorphisms of ${}^\epsilon\mathbb{C}_2$ via the automorphisms of $C\!\ell_{1,3}$. In accordance with Theorem 9 the automorphism group of $C\!\ell_{1,3}$ is isomorphic to the Abelian group $\mathbb{Z}_4$. The non-Cliffordian group double covering the Lorentz group $O(1,3)$ has a form (theorem 10)

$$\begin{aligned}\mathbf{Pin}^{-,-,+}(1,3) &\simeq \frac{(\mathbf{Spin}_0(1,3)\odot\mathbb{Z}_2\otimes\mathbb{Z}_4)}{\mathbb{Z}_2}\\ &\simeq \frac{(SL(2,\mathbb{C})\odot\mathbb{Z}_2\otimes\mathbb{Z}_4)}{\mathbb{Z}_2}.\end{aligned}$$

In the group $\mathsf{A}_- = \{\mathsf{I},\mathsf{W},\mathsf{E},\mathsf{C}\}$ the matrices E and C of the antiautomorphisms $\mathcal{A}\rightarrow\widetilde{\mathcal{A}}$ and $\mathcal{A}\rightarrow\widetilde{\mathcal{A}^\star}$ for the basis (43) have the form

$$\mathsf{E}=\begin{pmatrix}0&1&0&0\\-1&0&0&0\\0&0&0&1\\0&0&-1&0\end{pmatrix},\quad \mathsf{C}=\begin{pmatrix}0&0&0&-i\\0&0&i&0\\0&-i&0&0\\i&0&0&0\end{pmatrix}$$

Further, in accordance with (21) an action of the antiautomorphism $\mathcal{A}\rightarrow\widetilde{\mathcal{A}^\star}$ on the spinor ϕ, defined by the matrix representation (45), is expressed as follows

$$\widetilde{\phi^\star}=\begin{pmatrix}\phi_1^* & \phi_2^* & -\phi_3^* & -\phi_4^*\\ -\phi_2 & \phi_1 & -\phi_4 & \phi_3\\ -\phi_3^* & -\phi_4^* & \phi_1^* & \phi_2^*\\ -\phi_4 & \phi_3 & -\phi_2 & \phi_1\end{pmatrix} \tag{47}$$

It should be noted that the same result may be obtained via $\Gamma^0\phi^+\Gamma^0$ (see [VR93]). Let us assume now that a massless Dirac-Hestenes equation (46) describes the neitrino field

$$\partial^\epsilon\phi\Gamma_{21}=0, \tag{48}$$

then the antineitrino field is described by an equation

$$(\widetilde{\partial^\epsilon\phi\Gamma_{21}})^\star=0. \tag{49}$$

The equations (48) and (49) are converted into each other under action of the antiautomorphism $\mathcal{A}\rightarrow\widetilde{\mathcal{A}^\star}$ which in accordance with Proposition 1 corresponds to the combination PT (charge conjugation by CPT–theorem). The fields describing by the equations (48) and (49) possess a fixed helicity (there exist no right neitrino and left antineitrino), since the automorphism $\mathcal{A}\rightarrow\mathcal{A}^\star$ corresponded to

parity reversal P in this case is not defined. On the other hand, in accordance with the Feynman-Stueckelberg interpretation the antiparticles are considered as particles moving back in time, therefore a time-reversed equation (48) describes the antiparticle (antineitrino). Moreover, in the Feynman–Stueckelberg interpretation time reversal for the chiral field gives rise to the well–known CP–invariance in the theory of weak interactions. Thus, the actions of the antiautomorphisms $\mathcal{A} \to \widetilde{\mathcal{A}}$ and $\mathcal{A} \to \widetilde{\mathcal{A}^\star}$ and the corresponding operations T and $PT \sim C$ on the field ${}^\epsilon\phi$ are equivalent. This equivalence immediately follows from (9) and (10) (see also [FRO90b]), namely, for $\mathcal{A} \in C\ell^+_{p,q}$ we have always $\widetilde{\mathcal{A}} = \widetilde{\mathcal{A}^\star}$, in our case ${}^\epsilon\phi \in C\ell^+_{1,3}$ and $\widetilde{{}^\epsilon\phi} = \widetilde{{}^\epsilon\phi^\star}$.

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References

[AlCh94] L.J. Alty, A. Chamblin, *Spin Structures on Kleinian Manifolds*, Class. Quantum Grav. **11**, 2411-2415, (1994).

[AlCh96] L.J. Alty, A. Chamblin, *Obstructions to Pin Structures on Kleinian Manifolds*, J. Math. Phys. **37**, 2001-2011 , (1996).

[Amm98] B. Ammann, **Spin-Strukturen und das Spektrum des Dirac-Operators** (Dissertation Freiburg 1998, Shaker-Verlag Aachen 1998).

[AtBSh] M.F. Atiyah, R. Bott, A. Shapiro, *Clifford modules*, Topology, **3**, (Suppl. 1),3-38, (1964).

[Bär91] C. Bär, **Das Spektrum von Dirac-Operatoren** (Dissertation, Bonner Math. Schriften **217**, 1991).

[Bau81] H. Baum, **Spin-Strukturen und Dirac-Operatoren über pseudoriemannschen Mannigfaltigkeiten** (Teubner, Leipzig, 1981).

[BLP89] V.B. Berestetskii, E.M. Lifshitz, L.P. Pitaevskii, **Quantum electrodynamics** (Nauka, Moscow 1989) [in Russian].

[BD89] M. Blau, L. Dąbrowski, *Pin structures on manifolds quotiented by discrete groups*, J. Geometry and Physics **6**, 143-157, (1989).

[BH] A. Borel and F. Hirzebruch, *Characteristic classes and homogeneous spaces*, Amer. J. Math. **80**, 458-538, **81** 315-382 and **82** 491-504 (1958, 1959 and 1960).

[CGT95] M. Cahen, S. Gutt and A. Trautman, *Pin structures and the modified Dirac operator*, J. Geometry and Physics **17**, 283-297 (1995).

[Ch94a] A. Chamblin, *Some Applications of Differential Topology in General Relativity*, J. Geometry and Physics **13**, 357-377 (1994).

[Ch94b] A. Chamblin, *On the Obstructions to Non-Cliffordian Pin Structures*, Commun. Math. Phys. **164**, 67-87, (1994).

[Ch97] A. Chamblin, *On the Superselection Sectors of Fermions*, preprint DAMTP R-97/4; hep-th/9704099, (1997).

[Che54] C. Chevalley, **The Algebraic Theory of Spinors** (Columbia University Press, New York, 1954).

[Che55] C. Chevalley, *The construction and study of certain important algebras*, Publications of Mathematical Society of Japan No 1 (Herald Printing, Tokyo, 1955).

[CF96] J.S.R. Chisholm, R.S. Farwell, *Properties of Clifford Algebras for Fundamental Particles*, in **Clifford (Geometric) Algebras**, ed. W. Baylis (Birkhuser, 1996), pp. 365–388.

[CDD82] Y. Choquet-Bruhat, C. De Witt-Morette and M. Dillard-Bleick, **Analysis, Manifolds and Physics** (North-Holland Publ. Co., Amsterdam, 1982).

[Cliff] W. K. Clifford, *Applications of Grassmann's extensive algebra*, Amer. J. Math. **1**, 350, (1878).

[Cor84] J.F. Cornwell, **Group Theory in Physics** (Academic Press, 1984).

[Cru90] A. Crumeyrolle, **Orthogonal and Symplectic Clifford Algebras. Spinor Structures** (Kluwer Acad. Publ., Dordrecht, 1990).

[Dąb88] L. Dąbrowski, **Group Actions on Spinors** (Bibliopolis, Naples, 1988).

[DP86] L. Dąbrowski, R. Percacci, *Spinors and Diffeomorphisms*, Commun. Math. Phys. **106**, 691-704, (1986).

[DP87] L. Dąbrowski, R. Percacci, *Diffeomorphisms, orientations, and pin structures in two dimensions*, J. Math. Phys. **29**, 580-593, (1987).

[DR89] L. Dąbrowski, M. Rinaldi, *Spin structures, spinors and the Dirac operator: real versus complex manifolds*, J. Geometry and Physics **6**, 651-656 (1989).

[DT86] L. Dąbrowski, A. Trautman, *Spinor structures on spheres and projective spaces*, J. Math. Phys. **27**, 2022-2088, (1986).

[DW90] B. De Witt, *The Pin Groups in Physics*, Phys. Rev. **D41**, p. 1901 (1990).

[DWGK] C. De Witt-Morette, Shang-Jr Gwo, E. Kramer, *Spin or Pin?*, preprint (1997); http://www.rel.ph.utexas.edu/Members/cdewitt/SpinOrPin1.ps.

[FRO90a] V.L. Figueiredo, W.A. Rodrigues, Jr., E.C. Oliveira, *Covariant, algebraic, and operator spinors*, Int. J. Theor. Phys. **29**, 371-395, (1990).

[FRO90b] V.L. Figueiredo, W.A. Rodrigues, Jr., E.C. Oliveira, *Clifford algebras and the hidden geometrical nature of spinors*, Algebras, Groups and Geometries **7**, 153-198, (1990).

[Fr97] Th. Friedrich, **Dirac-Operatoren in der Riemannschen Geometrie** (Vieweg-Verlag Braunschweig/Wiesbaden, 1997).

[Fr98] Th. Friedrich, *Cartan Spinor Bundles on Manifolds*, preprint SFB 288, N 303, TU-Berlin (1998).

[FT96] Th. Friedrich, A. Trautman, *Clifford structures and spinor bundles*, preprint SFB 288, N 251, TU-Berlin, (1996).

[FT99] Th. Friedrich, A. Trautman, *Spin spaces, Lipschitz groups, and spinor bundles*, preprint SFB 288, N 362, TU-Berlin (1999).

[Ger68] R.P. Geroch, *Spinor Structure of Spacetimes in General Relativity: I*, J. Math. Phys. **9**, 1739-1744 (1968).

[GP82] W. Greub, H.R. Petry, *Spinor structures*, Lect. Notes Math. **905**, 170-185 (1982).

[Hae56] A. Haefliger, *Sur l'extension du groupe structural d'un espace fibre*, C.R. Acad. Sci. Paris. **243**, 558-560 (1956).

[Hest66] D. Hestenes, **Space-Time Algebra** (Gordon & Breach, New York, 1966).

[Hest67] D. Hestenes, *Real spinor fields*, J. Math. Phys. **8**, 798-808, (1967).

[Hest76] D. Hestenes, *Observables, operators, and complex numbers in the Dirac theory*, J. Math. Phys. **16**, 556-571, (1976).

[Hest90] D. Hestenes, *The Zitterbewegung Interpretation of Quantum Mechanics*, Found. Phys. **20**, 1213-1232 (1990).

[Ish78] C.J. Isham, *Spinor fields in four dimensional space-time*, Proc. R. Soc. Lond. A. **364**, 591-599 (1978).

[Kar68] M. Karoubi, *Algebres de Clifford et K-theorie*, Ann. scient. Ec. Norm. Sup. 4^e ser. t.1, 161-270 (1968).

[Kar79] M. Karoubi, **K-Theory. An Introduction** (Springer-Verlag, Berlin, 1979).

[KT89] R.C. Kirby, L.R. Taylor, *Pin Structures on Low-Dimensional Manifolds*, London Math. Soc. Lecture Notes, no. 151, CUP, (1989).

[Lips] R. Lipschitz, **Untersuchungen über die Summen von Quadraten** (Max Cohen und Sohn, Bonn, 1886).

[Lou81] P. Lounesto, *Scalar Products of Spinors and an Extension of Brauer-Wall Groups*, Found. Phys. **11**, 721-740, (1981).

[Lou93] P. Lounesto, *Clifford algebras and Hestenes spinors*, Found. Phys. **23**, 1203-1237 (1993).

[Lou96] P. Lounesto, *Counter-Examples in Clifford Algebras*, Advances in Applied Clifford Algebras **6**, 69-104 (1996).

[LM89] H.B. Lowson, M.-L. Michelsohn, **Spin Geometry** (Princeton University Press, Princeton 1989).

[Mil63] J. Milnor, *Spin structures on manifolds*, Enseign. Math. **9**, 198-203 (1963).

[Port] I. Porteous, **Topological Geometry** (van Nostrand, London, 1969).

[Ras55] P.K. Rashevskii, *The Theory of Spinors*, (in Russian) Uspekhi Mat. Nauk **10**, 3-110 (1955); translation in Amer. Math. Soc. Transl. (Ser.2) **6**, 1 (1957).

[RRSV95] D.L. Rapoport, W.A. Rodrigues, Jr., Q.A.G. de Souza, and J. Vaz, Jr., *The Riemann-Cartan-Weyl geometry generated by a Dirac-Hestenes spinor field*, Algebras, Groups and Geometries **11**, 23-35 (1995).

[RF90] W.A. Rodrigues, Jr. and V.L. Figueiredo, *Real spin-Clifford bundle and the spinor structure of the spacetime*, Int. J. Theor. Phys. **29**, 413-424 (1990).

[RO90] W.A. Rodrigues, Jr. and E.C. Oliveira, *Dirac and Maxwell equations in the Clifford and spin-Clifford bundles*, Int. J. Theor. Phys. **29**, 397-412 (1990).

[RS93] W.A. Rodrigues, Jr. and Q.A.G. de Souza, *The Clifford bundle and the nature of the gravitational field*, Found. Phys. **23**, 1465-1490 (1993).

[RVR93] W.A. Rodrigues, Jr., J. Vaz, Jr. and E. Recami, *About Zitterbewegung and electron structure*, Phys. Lett. **B 318**, 623-628 (1993).

[RSVL] W.A. Rodrigues, Jr., Q.A.G. de Souza, J. Vaz, Jr., P. Lounesto, *Dirac-Hestenes spinor fields in Riemann-Cartan spacetime*, Int. J. Theor. Phys., **35**, 1849-1900, (1996).

[Sal81a] N. Salingaros, *Realization, extension, and classification of certain physically important groups and algebras*, J. Math. Phys. **22**, 226-232, (1981).

[Sal81b] N. Salingaros, *Algebras with three anticommuting elements. II. Two algebras over singular field*, J. Math. Phys. **22**(10), (1981).

[Sal82] N. Salingaros, *On the classification of Clifford algebras and their relation to spinors in n dimensions*, J. Math. Phys. **23**(1), (1982).

[Sal84] N. Salingaros, *The relationship between finite groups and Clifford algebras*, J. Math. Phys. **25**, 738-742, (1984).

[Sch49] J.A. Schouten, *On the geometry of spin spaces*, Indag. Math. **XI**, 3,4,5, (1949).

[Tr92] A. Trautman, *Spinors and the Dirac operator on hypersurfaces. I. General Theory*, J. Math. Phys. **33**, 4011-4019, (1992).

[Var99a] V.V. Varlamov, *Generalized Weierstrass representation for surfaces in terms of Dirac-Hestenes spinor field*, J. Geometry and Physics **32**(3), 241-251 (1999).

[Var99b] V.V. Varlamov, *On spinor fields on the surfaces of revolution*, Proc. Int. Conf. "Geometrization of Physics IV", Kazan State University, Kazan, October 4–8, 1999, pp. 248–253.

[VR93] J. Vaz, Jr. and W.A. Rodrigues, Jr., *Equivalence of the Dirac and Maxwell equations and quantum mechanics*, Int. J. Theor. Phys. **32**, 945-958 (1993).

[Wh78] G.S. Whiston, *Compact Spinor Spacetimes*, J. Phys. A: Math. Gen. **11**, No. 7 (1978).